# The Invertible Condensed Phase Spectrum Program: A Conditional Synthesis of Locality, Gaps, Phases, and Realization

Matthew Long
*The YonedaAI Collaboration*
*YonedaAI Research Collective*
Chicago, IL
matthew@yonedaai.com · https://yonedaai.com

14 July 2026

**Abstract**

This paper asks whether local quantum interactions can be organized into an invertible condensed phase spectrum without erasing the analytic hypotheses that make a lattice model physical. The proposed construction has five stages:

local quantum interactions
↓
condensed moduli stack of Hamiltonians
↓
uniformly gapped propositional substack
↓
stabilized phase $\infty$-groupoid
↓
invertible condensed phase spectrum.

The sequence is a research program, not a chain of established equivalences. Its analytic input is solid in several controlled settings. Common interaction norms give Lieb-Robinson bounds. Family-uniform thermodynamic dynamics additionally require a common interaction tail that vanishes at large distance. Continuous parameter families in a fixed C*-algebra preserve positivity and the C*-norm. Uniform finite-volume or GNS gaps support quasi-adiabatic spectral flow and phase stability. Functional calculus maps controlled free families to operator K-theory. Low-dimensional Kitaev, BDI, and AKLT models provide explicit realizability witnesses.

The categorical assembly is less complete. A quantitative gap witness is data, so its projection to Hamiltonians is a fibration and not a subobject. The qualitative gapped locus is instead the propositional image of that projection. Localization by controlled gapped paths, stabilization by atomic ancillas, stacking, and passage to the

invertible sector should then produce a grouplike symmetric monoidal condensed anima. A connective spectrum follows if these constructions satisfy descent and the required coherences. Those hypotheses remain proposed or conjectural.

We keep the resulting object distinct from operator K-theory, Kubota's microscopic Omega-spectrum, and the Freed-Hopkins bordism-based field-theory classification. No identification is made without a comparison theorem. For a family on $B$ with gapless locus $\Sigma$ and $U = B \setminus \Sigma$, a class $\nu \in E^q(U)$ has the established relative boundary $\partial\nu \in E^{q+1}(B, U)$. Its interpretation as a physical transition charge is conditional on a microscopic comparison map. Excision and an $E$-oriented normal bundle of codimension $c$ identify it with a class in $E^{q+1-c}(\Sigma)$; without orientation the target is twisted.

The result is a precise conditional assembly theorem, a claim-status ledger, and an executable Haskell validator that rejects the principal false inferences. It does not claim a completed condensed spectrum or a universal classification of topological matter.

# Contents

# 1 The research question

Topological phases are commonly described by stable labels. Depending on the problem, the label may be a Chern number, a K-class, a bordism invariant, or a homotopy group of a space of gapped systems. A label is useful because it survives a controlled deformation. It is dangerous when the deformation rules and the comparison map that produced it are left implicit.

The condensed-mathematics perspective begins one step earlier [6]. It asks for a moduli object whose test points are parameterized families of local Hamiltonians. Profinite tests are especially natural for disorder hulls, inverse limits, and totally disconnected parameter spaces. Descent should glue compatible local descriptions, while higher morphisms should remember paths, homotopies of paths, pumps, and defects.

The program studied here can be stated in one sentence.

> Construct a condensed moduli object of uniformly local Hamiltonian families, cut out its uniformly gapped qualitative locus, impose the physical phase relation and stabilization, and spectrify the stacking-invertible sector.

Every verb in that sentence hides a theorem obligation. "Construct" asks for a site, objects, morphisms, and descent. "Cut out" asks whether a gap condition is stable under restriction and gluing. "Impose" asks for a calculus of controlled paths and quasi-local equivalences. "Stabilize" asks which ancillas count as trivial. "Spectrify" asks for coherent symmetric monoidal structure and deloopings. The paper keeps those obligations visible.

### 1.1 Five stages, not five equivalences

Write the proposed sequence as

$$\mathsf{Int}^{\mathrm{ub}}_{\mathcal{F}} \longrightarrow \mathfrak{Ham}_{\mathcal{F}} \longleftarrow \mathfrak{Gap}^{\mathrm{prop}}_{\mathcal{F}} \longrightarrow \mathfrak{Phase}^{\mathrm{st}}_{\mathcal{F}} \longrightarrow IP^{\mathrm{cond}}_{\mathcal{F}}. \tag{1}$$

The reverse arrow at the gapped stage is intentional: a gapped subobject, if it exists, includes into the Hamiltonian moduli object. A quantitative witness object maps to both. The prose sequence records a flow of construction; the categorical diagram records the variance correctly.

No arrow in equation (1) is declared an equivalence. The first packages analytic data. The second takes a propositional image. The third localizes and stabilizes. The fourth passes to stacking-invertible objects and then, conditionally, to a spectrum.

### 1.2 Spatial and spacetime grading

We use

$$d = \text{spatial dimension}, \qquad n = d + 1 = \text{spacetime dimension}. \tag{2}$$

Microscopic interactions and Kubota's construction are graded by $d$. Field-theory tangential structures are graded by $n$. Any comparison must display the shift. Suppressing it can make two formulas look identical even when they classify different dimensional objects.

### 1.3 Scope

The target is the invertible sector. Intrinsic topological order with noninvertible excitations needs fusion, braiding, and defect categories that cannot be recovered from a spectrum of units. Crystalline symmetry, fermionic grading, and disorder may be included only after their actions are specified in the microscopic category. This paper describes interfaces for that work rather than claiming a complete classification.

## 2 A status grammar for a mixed theorem program

The source papers combine imported theorems, finite calculations, definitions, and proposed categorical constructions. A single word such as "result" cannot distinguish them. We use five statuses.

**Definition 2.1** (Claim status)**.** For a statement made under explicit hypotheses:

- Established means a theorem or complete calculation is available in the stated setting.
- Proposed means a definition or construction is specified, but its full existence, functoriality, or usefulness has not been proved.
- Conjectural means a mathematically definite statement is expected to be true and lacks a proof.
- Open means the paper asserts neither a construction nor a resolution.

- OBSTRUCTED means the stated implication is false or is ruled out under the stated rules.

The difference between OPEN and OBSTRUCTED matters. A missing lattice model is not an obstruction. An anomaly theorem or an explicit interacting counterexample may be one. The difference between PROPOSED and CONJECTURAL also matters. The notation $IP^{\mathrm{cond}}$ is a proposed target. A claim that it agrees with another spectrum on a specified overlap is conjectural only after a comparison map and its expected domain have been stated.

## 2.1 An evidence order

Statuses do not form a simple scale from weak to strong. An obstructed implication can be known more firmly than a proposed construction. For validation purposes it is better to record provenance separately:

$$\text{claim} \quad + \quad \text{status} \quad + \quad \text{hypotheses} \quad + \quad \text{provider}. \tag{3}$$

A provider may be a theorem, an explicit calculation, a cited construction, or a named open comparison problem. The Haskell code mirrors equation (3); it never tries to infer truth from a status label.

## 2.2 Nonclaims fixed at the outset

The following statements are excluded.

1. A profinite parameter space does not imply a uniform locality bound.
2. Pointwise finite-volume gaps do not imply a thermodynamic gap.
3. A ring with involution is not automatically a C*-algebra.
4. A K-class does not reconstruct a Hamiltonian.
5. A low-energy field theory is not automatically equivalent to its microscopic lattice model.
6. Isomorphic homotopy groups do not identify spectra.
7. An abstract bordism class does not provide a microscopic realization.
8. A relative cohomology class is not automatically a measurable transition charge.

These exclusions are not philosophical caution. Each one corresponds to a missing hypothesis or a known loss of information.

# 3 Stage one: local quantum interactions

Let $\Gamma$ be a countable metric lattice of bounded geometry. At each site $x$ choose a finite-dimensional Hilbert space $\mathcal{H}_x$, or a finite dimensional graded local algebra for fermions. The quasi-local algebra $\mathcal{A}_\Gamma$ is the norm completion of the directed union of finite-region matrix algebras.

An interaction is a map

$$\Phi : \{X \Subset \Gamma\} \longrightarrow (\mathcal{A}_X)_{\mathrm{sa}}. \tag{4}$$

Choose a decay function $F : [0, \infty) \to (0, \infty)$ with the summability and convolution properties used in Lieb-Robinson theory. One standard norm has the form

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \frac{1}{F(d(x, y))} \sum_{X \ni x,y} \|\Phi(X)\|. \tag{5}$$

Other weighted norms are possible. What matters in a family is that one norm convention and one bound work for every parameter.

**Definition 3.1** (Uniformly bounded local family)**.** For a compact or profinite parameter object $S$, a family $\Phi = (\Phi_s)_{s \in S}$ is uniformly $F$-local when

$$\sup_{s \in S} \|\Phi_s\|_F < \infty \tag{6}$$

and its local coefficients $s \mapsto \Phi_s(X)$ are norm-continuous for every finite $X$. When infinite tails are admitted, the family also carries an explicit tail bound uniform in $s$.

Local coefficient continuity and equation (6) are independent conditions. A family can vary continuously on every finite region while its interaction range escapes to infinity. Conversely, a uniformly finite-range assignment can jump discontinuously in its coefficients.

## 3.1 Established locality output

**Theorem 3.2** (Uniform Lieb-Robinson control)**.** ESTABLISHED *Suppose a family of interactions has a common finite $F$-norm on a bounded-geometry lattice. Then the finite-volume dynamics satisfy a Lieb-Robinson estimate with constants that can be chosen uniformly over the family. For local observables $A$ and $B$ with separated supports, one obtains a bound of the form*

$$\|[\tau_t^{\Lambda,s}(A), B]\| \leq C\|A\|\|B\|\big(e^{v|t|} - 1\big) \sum_{x \in \operatorname{supp} A} \sum_{y \in \operatorname{supp} B} F(d(x, y)), \tag{7}$$

*where $C$ and $v$ do not depend on $s$ or on the finite volume $\Lambda$ [12, 13]. For the thermodynamic limit of the full family, require separately a common tail function $T_F(R)$ with $T_F(R) \to 0$ as $R \to \infty$. Finite-region $F$-norm continuity does not imply this uniform tail condition.*

This theorem is analytic input, not a consequence of condensed descent. A test object organizes the parameter dependence; it does not manufacture the common constant in equation (6).

### 3.2 Restriction and finite descent

For a map $f : T \to S$, restriction sends $\Phi$ to $f^*\Phi$. The same uniform locality bound remains valid. For a finite jointly surjective cover $S_i \to S$ of profinite spaces, compatible local coefficient functions glue because the disjoint union map is a quotient map. A common numerical bound also glues if it has been supplied on the entire finite cover, for example by taking the maximum of finitely many bounds.

This gives a controlled set-valued sheaf of interaction objects on the chosen finite-cover site. It does not yet give the higher moduli stack used later. Morphisms, automorphisms, and higher coherences still have to be defined.

**Counterexample 3.3** (Moving weak defect)**.** Let $S = \mathbb{N} \cup \{\infty\}$ be the one-point compactification. A family may place a weak local perturbation farther from the origin as the parameter approaches $\infty$. Every fixed local coefficient then converges, and the interactions may share a locality norm. A low-energy mode tied to the moving defect can nevertheless make the spectral gap tend to zero. Local continuity and uniform locality therefore do not imply a uniform gap.

## 4 Stage two: a condensed Hamiltonian moduli object

The interaction sheaf records objects. A moduli stack must also record isomorphisms and higher families. Its definition depends on which changes of microscopic presentation count as equivalences.

### 4.1 Objectwise C*-algebraic control

For a compact Hausdorff parameter space $S$ and a C*-algebra $\mathcal{A}$, the continuous family algebra

$$C(S, \mathcal{A}) \tag{8}$$

is a C*-algebra with pointwise operations, involution, and norm

$$\|a\| = \sup_{s \in S} \|a(s)\|_{\mathcal{A}}. \tag{9}$$

Positivity is fiberwise:

$$a \geq 0 \quad \Longleftrightarrow \quad a(s) \geq 0 \text{ for every } s \in S. \tag{10}$$

The C*-identity follows exactly:

$$\|a^*a\| = \sup_s \|a(s)^*a(s)\| = \sup_s \|a(s)\|^2 = \|a\|^2. \tag{11}$$

These statements solve the positivity and norm problem once the fixed C*-algebra $\mathcal{A}$ has been chosen. They do not construct that norm from bare algebraic multiplication and involution.

**Counterexample 4.1** (A star algebra can have several C*-completions)**.** A dense involutive convolution algebra associated with a nonamenable group can admit distinct full and reduced C*-norms. Its multiplication and star operation do not select one completion. Any Hamiltonian moduli problem that uses spectra, positivity, or states must carry the completion as part of its input.

## 4.2 Finite-cover C*-descent

If $S_i \to S$ is a finite jointly surjective family of profinite spaces, then

$$C(S, \mathcal{A}) \cong \mathrm{Eq}\left(\prod_i C(S_i, \mathcal{A}) \rightrightarrows \prod_{i,j} C(S_i \times_S S_j, \mathcal{A})\right) \tag{12}$$

as C*-algebras. The norm on a glued section is forced by the fiberwise supremum. Positivity and self-adjointness descend because they are tested fiberwise.

Finite clopen partitions give dense finite approximations to continuous functions on a profinite space. They do not imply that every continuous family factors exactly through a finite quotient. That stronger assertion would discard genuine continuous variation.

## 4.3 States are not pointwise state-valued maps

A state on $C(S, \mathcal{A})$ contains a probability measure on $S$ together with conditional state data. In general,

$$\mathrm{State}(C(S, \mathcal{A})) \neq C(S, \mathrm{State}(\mathcal{A})). \tag{13}$$

The right side chooses one normalized state at every point. The left side can average over the parameter space. This distinction is relevant when a ground-state sector is part of a gap witness.

## 4.4 The proposed moduli stack

Let $\mathsf{CondTest}$ be a chosen site of condensed test objects. Its precise size, topology, and hypercompletion convention must be fixed. For each $S$, define a tentative category $\mathcal{C}_{\mathfrak{Ham}}(S)$ whose objects include:

1. a lattice or relaxed-lattice datum of spatial dimension $d$;
2. a continuous field of local finite-dimensional algebras;
3. a uniformly $F$-local self-adjoint interaction;
4. symmetry and grading data;
5. boundary and ground-state conventions when required.

Morphisms should record controlled changes of presentation, such as local unitary identifications, lattice refinements, or paths of interactions. The choice determines the eventual phase relation.

**Definition 4.2** (Condensed Hamiltonian moduli, proposed). PROPOSED The object $\mathfrak{Ham}^G_{d,\mathcal{F}}$ is the hypercomplete sheafification of the anima-valued presheaf

$$S \longmapsto \mathrm{Core}(\mathcal{C}_{\mathfrak{Ham}}(S)), \tag{14}$$

provided restriction preserves the analytic bounds and the chosen morphisms satisfy descent.

The word "provided" carries real content. Objectwise finite-cover descent does not automatically prove descent for path spaces, automorphism groups, spectral-flow providers, or variable lattices. The existence of equation (14) with all desired structures is PROPOSED.

# 5 Stage three: the uniformly gapped propositional substack

A thermodynamic gap is quantitative. A gapped locus is qualitative. The two are related by a projection, but they are not the same object.

## 5.1 Finite-volume and GNS witnesses

Fix a family of finite volumes $\Lambda_L$ and a prescribed low-energy sector of dimension $m_L$. A uniform finite-volume witness contains a number $\gamma > 0$ and a threshold $L_0$ such that

$$E_{m_L+1}(H_{\Lambda_L,s}) - E_{m_L}(H_{\Lambda_L,s}) \geq \gamma \tag{15}$$

for every $L \geq L_0$ and every parameter $s$.

An infinite-volume GNS witness instead chooses a ground state $\omega_s$, its GNS representation, and a positive generator $K_s$ of the implemented dynamics such that

$$\mathrm{Spec}(K_s) \cap (0, \gamma) = \varnothing \tag{16}$$

uniformly in $s$. Relating the two notions requires hypotheses on boundary conditions, ground-state sectors, and the thermodynamic limit. The paper does not treat them as definitionally equal.

## 5.2 Witnesses form a fibration of data

Let $W(S)$ denote the type of accepted gap witnesses over an $S$-family. Define

$$\mathfrak{Gap}^{\mathrm{wit}}_{\mathcal{F}}(S) = \{(H, w) : H \in \mathfrak{Ham}_{\mathcal{F}}(S),\ w \in W_S(H)\}. \tag{17}$$

Forgetting the numerical and ground-state data gives

$$p_S : \mathfrak{Gap}^{\mathrm{wit}}_{\mathcal{F}}(S) \longrightarrow \mathfrak{Ham}_{\mathcal{F}}(S). \tag{18}$$

**Proposition 5.1** (Witness projection is not a subobject). *Even when all restriction maps are defined, $p_S$ need not be a monomorphism. One Hamiltonian family can admit many gap lower bounds, thresholds, ground-state presentations, and equivalent proofs. Thus $\mathfrak{Gap}^{\mathrm{wit}}$ is a fibration of quantitative data over the Hamiltonian moduli object, not the qualitative gapped substack.*

This is more than a set-theoretic nuisance. Loops in the space of witnesses can contain information about choices that should disappear from a phase label. Treating equation (18) as an inclusion would retain that information by mistake.

### 5.3 The propositional image

The qualitative predicate is obtained by truncating the existence of a witness:

$$\mathfrak{Gap}^{\mathrm{prop}}_{\mathcal{F}}(S) = \{H \in \mathfrak{Ham}_{\mathcal{F}}(S) : \|\exists w \in W_S(H)\|\}. \tag{19}$$

Here the vertical bars denote propositional truncation. Equivalently, $\mathfrak{Gap}^{\mathrm{prop}}$ is the image of $p$ when the chosen category of condensed anima has pullback-stable image factorizations.

**Definition 5.2** (Uniformly gapped propositional substack, proposed)**.** Proposed If gap witnesses restrict functorially, compatible witnesses glue over the chosen covers with a common positive lower bound, and the image factorization in equation (19) commutes with pullback, then $\mathfrak{Gap}^{\mathrm{prop}}_{\mathcal{F}} \hookrightarrow \mathfrak{Ham}_{\mathcal{F}}$ is the uniformly gapped propositional substack.

Descent can combine witnesses that already exist. It cannot infer a common positive lower bound from unrelated local or pointwise gaps. For a finite cover, finitely many compatible lower bounds have a positive minimum. For an infinite cover, the infimum can be zero.

### 5.4 Gap stability

**Theorem 5.3** (Controlled stability, schematic)**.** Established *in its cited operator-algebraic settings. Let $r \mapsto \Phi_r$ be a path of interactions with common locality bounds and a uniform thermodynamic gap. Under the standard differentiability and decay hypotheses, quasi-adiabatic spectral flow gives quasi-local automorphisms that transport the ground-state sectors along the path. Phase invariants defined through this transport remain constant [4, 3].*

The theorem begins with a uniform gap. It does not prove that a proposed path is gapped. Establishing the gap is often the hardest analytic task in a phase classification.

## 6 Stage four: the stabilized phase infinity-groupoid

The qualitative gapped locus still contains every microscopic presentation. A phase object should identify systems connected by the allowed physical moves.

### 6.1 The path category

For a test object $S$, let $\mathcal{C}_{\mathrm{gap}}(S)$ be the proposed category whose objects are $S$-families in $\mathfrak{Gap}^{\mathrm{prop}}(S)$ and whose morphisms are controlled uniformly gapped interpolations, quasi-local automorphisms supplied by spectral flow, and specified changes of local presentation. Every morphism carries a provider showing that locality and the gap remain uniform over both $S$ and the interpolation parameter.

Let $W(S)$ be the class of morphisms declared to be phase equivalences. A calculus of fractions is not assumed automatically. The infinity-categorical localization

$$\mathcal{C}_{\mathrm{gap}}(S)[W(S)^{-1}] \tag{20}$$

exists abstractly in a sufficiently large ambient category, but size, descent, and compatibility with stacking must still be checked.

## 6.2 Stabilization

Choose a class $\mathsf{At}$ of atomic product-state systems. Stacking with an atomic ancilla gives transition functors

$$H \longmapsto H \boxtimes A, \qquad A \in \mathsf{At}. \tag{21}$$

The stabilized category is a filtered colimit only after maps between ancilla choices and their coherences have been specified. Formally, write

$$\mathcal{C}^{\mathrm{st}}_{\mathrm{gap}}(S) = \operatorname*{colim}_{A \in \mathsf{At}} \mathcal{C}_{\mathrm{gap}}(S)[W(S)^{-1}]. \tag{22}$$

**Definition 6.1** (Stabilized phase infinity-groupoid, proposed)**.** Proposed Under the existence and descent hypotheses above, set

$$\mathfrak{Phase}^{\mathrm{st}}_{d,\mathcal{F},G}(S) = \mathrm{Core}\big(\mathcal{C}^{\mathrm{st}}_{\mathrm{gap}}(S)\big). \tag{23}$$

The result is intended to be an anima-valued condensed sheaf.

Taking the core discards noninvertible categorical morphisms but retains all equivalences and their higher homotopies. It does not make every object invertible under stacking. That is a separate operation.

## 6.3 Why stabilization changes the question

Two finite-band Hamiltonians may be inequivalent before adding trivial bands and equivalent afterward. Two spin chains may become equivalent after stacking with a product state that changes the on-site Hilbert space. Stabilization is therefore a quotient choice, not a harmless notational convenience.

The BDI example goes further. Allowing symmetry-preserving interactions enlarges the path category and reduces the free integer classification to $\mathbb{Z}/8$ [7]. A stable free K-class does not survive unchanged when the microscopic morphisms are changed.

# 7 Stage five: the invertible condensed phase spectrum

Stacking supplies a symmetric monoidal product on microscopic systems. If it descends through the gap condition, localization, and stabilization, it makes $\mathfrak{Phase}^{\mathrm{st}}$ an $E_\infty$ object in condensed anima.

## 7.1 The invertible sector

An object $x$ is stacking-invertible when there is an object $y$ and an equivalence

$$x \boxtimes y \simeq \mathbf{1} \tag{24}$$

in the stabilized phase infinity-groupoid. The full subobject of such objects is

$$\mathrm{Pic}(\mathfrak{Phase}^{\mathrm{st}}). \tag{25}$$

It is grouplike if the monoidal coherences and descent are in place.

*Remark* 7.1 (Group completion and units). One may group-complete the entire stacking monoid and then study its units, or take the already invertible full subobject. These operations need not encode the same physical question if formal inverses exist without microscopic witnesses. The program here uses the Picard subobject: an inverse must be represented by a phase.

## 7.2 Spectrification

Grouplike $E_\infty$ spaces are equivalent to connective spectra. In a suitable infinity-topos, the same recognition principle can be applied internally, subject to presentability and descent hypotheses. This motivates the definition

$$IP^{\mathrm{cond}}_{d,\mathcal{F},G} = \mathrm{sp}\,\mathrm{Pic}(\mathfrak{Phase}^{\mathrm{st}}_{d,\mathcal{F},G}). \tag{26}$$

**Definition 7.2** (Invertible condensed phase spectrum, proposed)**.** PROPOSED Equation equation (26) denotes the connective condensed spectrum obtained from the stacking-invertible stabilized phase object, provided the latter exists as a grouplike $E_\infty$ condensed anima and the internal recognition theorem applies in the chosen category.

This definition does not prove that the spectrum exists for the full class of interacting lattice systems. It also does not identify its spatial degree $d$ pieces with the levels of an Omega-spectrum. A spatial suspension or lattice-direction delooping theorem is still required.

## 7.3 A conditional assembly theorem

The logical content of the program can be packaged as follows.

**Theorem 7.3** (Conditional assembly)**.** *Assume:*

*(A1) the uniformly $F$-local interaction presheaf has functorial restrictions and hyperdescent;*

*(A2) Hamiltonian objects, controlled morphisms, and higher equivalences form an anima-valued condensed stack;*

*(A3) quantitative finite-volume or GNS gap witnesses restrict and descend with a common positive lower bound;*

*(A4) the witness projection admits a pullback-stable propositional image;*

*(A5) controlled gapped paths and quasi-local equivalences admit a localization compatible with descent;*

*(A6) stabilization by the declared atomic systems exists and commutes with restriction;*

*(A7) stacking is coherently symmetric monoidal throughout the construction and preserves all declared equivalences;*

*(A8) the Picard subobject is a grouplike $E_\infty$ condensed anima to which internal connective spectrification applies.*

*Then the five stages define a connective invertible condensed phase spectrum $IP^{\mathrm{cond}}_{d,\mathcal{F},G}$. Its zeroth homotopy sheaf is the sheaf of stacking-invertible stabilized phases represented in the construction.*

*Proof.* Assumptions (A1) and (A2) give the condensed Hamiltonian moduli object. Assumptions (A3) and (A4) give a propositional gapped subobject without retaining choices of gap proof. Assumptions (A5) and (A6) produce the stabilized phase infinity-groupoid. Assumption (A7) equips it with a coherent $E_\infty$ stacking product. Assumption (A8) identifies the Picard subobject with the zeroth space of a connective spectrum internal to the chosen condensed setting. The description of the zeroth homotopy sheaf is the definition of connected components in that Picard object. □

The theorem is formally valid but conditional. Its conclusion should not be cited without the assumptions. Current work establishes parts of (A1), (A3), and (A5) in controlled settings. The full stack-level assembly is PROPOSED, and broad comparison or spatial delooping statements are OPEN or CONJECTURAL.

# 8 Homotopy groups, pumps, and defects

The slogan

$$\begin{aligned} \pi_0 &= \text{phases}, \\ \pi_1 &= \text{adiabatic pumps and phase automorphisms}, \\ \pi_k &= \text{higher families and higher defects} \end{aligned} \tag{27}$$

is useful only after the basepoint and the kind of homotopy group have been specified.

## 8.1 Homotopy sheaves and global shapes

For a based condensed anima $X$, one can form homotopy sheaves $\underline{\pi}_k(X)$ on test objects. One can also take global sections or a shape and then form ordinary homotopy groups. These operations need not agree. In the proposed spectrum, the least ambiguous object is the homotopy sheaf

$$\underline{\pi}_k\big(IP^{\mathrm{cond}}_{d,\mathcal{F},G}\big). \tag{28}$$

Its value at a test object $S$ records $S$-parameterized classes, subject to the success of the assembly theorem.

## 8.2 Zeroth homotopy

At a point test object,

$$\pi_0 \Omega^\infty IP^{\mathrm{cond}} \tag{29}$$

would be the abelian group of stacking-invertible stabilized phases. It is not the set of all topological phases, since noninvertible phases have been removed. It also depends on symmetry, allowed interactions, boundary conventions, and the selected stabilization.

## 8.3 First homotopy

A based loop in the phase infinity-groupoid is a one-parameter family that returns to its starting phase up to the declared equivalence. With a uniform gap and a quasi-adiabatic transport provider, such a loop can define an adiabatic pump or an automorphism of the phase. Without that analytic provider it is only a loop in the proposed moduli object.

Thus the identification of $\pi_1$ with pumps has two layers:

1. categorical loop data in the stabilized phase object;
2. an analytic transport theorem that assigns a physical pumped quantity or boundary action.

The first is part of the proposed organization. The second is established in important model classes and remains open in full generality.

## 8.4 Higher families

A class in $\pi_k$ can be represented by a based $k$-parameter family. Its interpretation as a codimension-$(k+1)$ defect requires a separate family-to-defect construction, usually involving suspension, boundary conditions, or a clutching map. Higher homotopy does not turn into a defect merely by changing terminology.

**Proposition 8.1** (Qualified higher-defect interpretation)**.** *Suppose a family-to-defect map is defined on controlled uniformly gapped families, respects homotopy and stacking, and is compatible with the chosen spatial suspension. Then a class in the relevant $\pi_k$ determines a stable defect class. Without this map, $\pi_k$ records a higher family but not yet a physical defect.*

# 9 The gapless locus and relative transition charge

Let $B$ be a parameter space and let

$$f : B \longrightarrow \mathfrak{Ham}_{\mathcal{F}} \tag{30}$$

be a Hamiltonian family. Pull back the propositional gapped subobject:

$$U_f = B \times_{\mathfrak{Ham}_{\mathcal{F}}} \mathfrak{Gap}_{\mathcal{F}}^{\mathrm{prop}}. \tag{31}$$

When $U_f$ is represented by an open subspace of $B$, define

$$\Sigma_f = B \setminus U_f. \tag{32}$$

The symbol $\Sigma$ therefore denotes a qualitative failure locus. The witness fibration also pulls back to $U_f$, but it cannot be subtracted from $B$ because it is not a subobject. If openness, closedness of $\Sigma$, or a good-pair replacement is unavailable, the relative group used below is not assigned automatically to the proposed gapless locus.

## 9.1 Why openness is an analytic theorem

For bounded self-adjoint elements, a spectral gap is stable under sufficiently small norm perturbations. For thermodynamic interacting systems, openness requires a topology on interactions and a stability theorem uniform in volume. The statement "the gapped locus is open" is therefore established only in a declared controlled setting. Condensed sheafification alone does not prove it.

## 9.2 The exact relative class

Let $E$ be a generalized cohomology theory represented by a spectrum. Put $U = B \setminus \Sigma$ and assume $(B, U)$ is a good pair for $E$. The long exact sequence contains

$$E^q(B) \xrightarrow{j^*} E^q(U) \xrightarrow{\partial} E^{q+1}(B, U) \longrightarrow E^{q+1}(B). \tag{33}$$

For $\nu \in E^q(U)$, define

$$Q_\Sigma(\nu) = \partial\nu \in E^{q+1}(B, B \setminus \Sigma). \tag{34}$$

**Proposition 9.1** (Extension obstruction)**.** ESTABLISHED *The class $Q_\Sigma(\nu)$ vanishes exactly when $\nu$ lies in the image of $E^q(B) \to E^q(U)$.*

*Proof.* This is exactness of equation (33) at $E^q(U)$. □

The proposition concerns extension of a cohomology class. It does not say that the Hamiltonian family extends across $\Sigma$ as a uniformly gapped family.

## 9.3 Excision and Thom localization

If excision applies near the closed set $\Sigma$, then

$$E^{q+1}(B, B \setminus \Sigma) \cong E^{q+1}(N, N \setminus \Sigma) \tag{35}$$

for a suitable neighborhood $N$. Suppose further that $B$ is smooth, $\Sigma \hookrightarrow B$ is a closed submanifold of codimension $c$, and its normal bundle $\mathcal{N}$ is $E$-oriented. A tubular neighborhood and the Thom isomorphism give

$$E^{q+1}(B, B \setminus \Sigma) \cong \widetilde{E}^{q+1}(\mathrm{Th}(\mathcal{N})) \cong E^{q+1-c}(\Sigma). \tag{36}$$

Without an $E$-orientation, the last group must be replaced by the appropriate twisted theory. For singular $\Sigma$, a stratified normal datum or supported-cohomology formulation is needed.

## 9.4 When the class is physical

Suppose a microscopic family $H$ on $U$ has a phase invariant $\nu_{\mathrm{mic}}(H)$. To obtain the $E$-cohomology class used above one needs a natural transformation

$$c_E : \nu_{\mathrm{mic}}(H) \longmapsto \nu_E(H) \in E^q(U) \tag{37}$$

that respects restriction, stacking, and controlled gapped homotopy.

**Theorem 9.2** (Conditional physical transition charge)**.** *If equation (37) is supplied and has the stated naturality, then*

$$Q_\Sigma(H) := \partial \nu_E(H) \tag{38}$$

*is invariant under the chosen microscopic phase relation on $U$. It is localized near $\Sigma$ by excision and, under the Thom hypotheses, has degree $q+1-c$ on the critical locus.*

The algebraic-topology part of this theorem is Established. The general microscopic comparison is Proposed or Open, depending on the model class. For a controlled free-fermion family, functional calculus can provide it. For arbitrary interacting systems, no universal map is asserted.

## 9.5 A local SSH test

For the chiral two-band SSH Hamiltonian, the off-diagonal function

$$q(k) = t_1 + t_2 e^{ik} \tag{39}$$

has winding one when $|t_1| < |t_2|$ and winding zero when $|t_1| > |t_2|$. At the isolated crossing $t_1 = t_2$, $k = \pi$, a small linking circle in the combined momentum and mass coordinates has degree of magnitude one. This is a controlled example of equation (34). A loop in the tuning parameter alone would miss the momentum coordinate and need not link the degeneracy [15, 16].

# 10 Microscopic systems and effective field theories

The phrase "microscopic to effective" covers three separate operations. They should be drawn as three arrows:

$$\begin{tikzcd}
\mathfrak{Ham}^{\mathrm{micro,gap}} \arrow[r, dashed, "L"] & \mathsf{EFT}^{\mathrm{inv}} \arrow[r, "I", "\text{established on a stated EFT domain}"'] & \mathsf{Class}^{\mathrm{stable}} \arrow[dll, dashed, "R"] \\
\mathsf{Real}_{\mathrm{lattice}} \arrow[u, "\text{forget witnesses}"] & &
\end{tikzcd} \tag{40}$$

Here $L$ is a low-energy limit, $I$ extracts a stable invariant on a specified effective-theory domain, and $R$ is a realization problem whose target contains explicit stabilized microscopic witnesses. The final vertical map forgets those witnesses and retains the underlying microscopic Hamiltonian. None is automatically inverse to another. The solid $I$ does not assert an invariant map on every object called an EFT; its established use here is restricted to declared targets, including the Freed-Hopkins domain described below.

### 10.1 Controlled free functional calculus

Let $h \in M_N(C(S, \mathcal{A}))_{\mathrm{sa}}$ and suppose

$$\mathrm{Spec}(h_s) \cap (-\gamma, \gamma) = \varnothing \tag{41}$$

for a common $\gamma > 0$. Continuous functional calculus gives

$$q_h = \mathrm{sgn}(h), \qquad p_h = \frac{1 - q_h}{2}. \tag{42}$$

The projection $p_h$ varies continuously with $s$. Relative to a reference $h_{\mathrm{ref}}$, the difference

$$\kappa(h, h_{\mathrm{ref}}) = [p_h] - [p_{h_{\mathrm{ref}}}] \tag{43}$$

is invariant under norm-continuous paths with a common gap and is additive under block sum. Throughout this paper the first argument is the target and the second is the reference, so $\kappa(\mathrm{target}, \mathrm{reference})$ means target minus reference.

This is an Established map in the controlled free sector. It loses the gap magnitude, dispersion, Fermi velocity, correlation length, Lieb-Robinson constants, coupling values, boundary termination, and much of the band presentation. Flattening is not a renormalization-group construction.

### 10.2 Interacting systems

A general interacting many-body Hamiltonian does not have a bounded one-particle matrix to which equation (42) applies. Flattening a finite-volume matrix can produce a highly nonlocal operator and may not have a quasi-local thermodynamic limit. Quasi-adiabatic spectral flow is the right comparison tool for paths of interacting systems, but it starts from uniform locality and a uniform gap.

The Fidkowski-Kitaev reduction is a decisive warning. One-dimensional BDI free phases have an integer index. Symmetry-preserving interactions reduce the classification to $\mathbb{Z}/8$ [7]. Therefore no universal equivalence can identify free operator K-theory with the interacting microscopic phase object.

### 10.3 Disorder

For a covariant disordered free Hamiltonian with a genuine C*-spectral gap, the Fermi projection lies in an appropriate crossed-product algebra and defines a K-class. A mobility-gap extension needs localization estimates strong enough to place the projection and its commutators in the relevant algebra. A profinite disorder hull does not supply those estimates.

## 11 Three stable-homotopy targets that remain distinct

The proposed condensed spectrum is not the only spectrum in the subject. Two established constructions have different domains.

## 11.1 Freed-Hopkins field-theory classes

Fix a spacetime symmetry type $(H_n, \rho_n)$ and its Madsen-Tillmann spectrum $MTH$. In the discrete topological sector, Freed and Hopkins compute deformation classes of reflection-positive invertible extended topological field theories by a torsion subgroup of

$$[MTH, \Sigma^{n+1} I\mathbb{Z}(1)]. \tag{44}$$

This is an Established field-theory theorem under its hypotheses [8]. Application to lattice phases uses an effective-field-theory assumption. The theorem does not assert that every class in equation (44) has a finite-range lattice representative with a thermodynamic gap.

## 11.2 Kubota's microscopic Omega-spectrum

Kubota constructs an Established Omega-spectrum $IP_*$ from invertible gapped quantum spin systems on Euclidean spaces. Its ingredients include operator-algebraic interactions, uniformly almost-local control, relaxed lattices, stabilization by atomic systems, nondegenerate gapped GNS ground states, and sheaves of smooth parameter families [11]. This is a genuine microscopic stable-homotopy object.

It is not definitionally the condensed spectrum in equation (26). The parameter sites, gap witnesses, lattice conventions, and descent conditions differ. A comparison should first be defined on their common smooth, uniformly controlled domain.

## 11.3 Aoki's K-theory bridge

Aoki proves, with connective and Bott-periodic qualifications, that solidification of algebraic K-theory of real or complex Banach algebras recovers the corresponding connective topological K-theory, followed by inversion of the degree-two complex or degree-eight real Bott element for the periodic theory. This is an Established invariant-level bridge [1].

The input Banach or C*-algebra is already present. Solidification does not recover its norm, positive cone, state space, or microscopic interaction. It also does not say that every K-class is a physical phase.

## 11.4 Comparison diagram

The honest current diagram is

$$\begin{tikzcd}
 & IP_*^{\mathrm{Kubota}} \arrow[dl, dotted, "C_{\mathrm{cond}}"'] \arrow[dr, dotted, "C_{\mathrm{EFT}}"] & \\
IP^{\mathrm{cond}} \arrow[rr, dotted, "C_{\mathrm{bord}}"'] & & \mathrm{Map}(MTH, \Sigma^{*+1} I\mathbb{Z}(1)) \\
 & K_{\mathrm{op}} \arrow[ul, dotted, "C_K"] \arrow[ur, dotted, "C_{K,\mathrm{EFT}}"'] &
\end{tikzcd} \tag{45}$$

Every dotted arrow needs a domain, a formula, naturality, stacking compatibility, and homotopy invariance. An equivalence claim also needs full faithfulness and essential surjectivity, or their spectral analogues. No such global equivalence is asserted here.

# 12 Physical realizability

An abstract class becomes physical only through a witness. For a microscopic system in spatial dimension $d$, a realization record contains:

1. local degrees of freedom and a lattice;
2. an explicit uniformly local interaction;
3. symmetry, grading, and boundary conventions;
4. a thermodynamic gap witness;
5. a microscopic invariant;
6. an explicit comparison to the abstract class;
7. a stacking inverse if invertibility is claimed.

An element of a bordism group supplies none of the microscopic fields by itself.

## 12.1 Three realized low-dimensional rows

The companion realizability analysis establishes three concrete rows.

| System | Gap witness | Microscopic invariant | Abstract comparison status |
|---|---|---|---|
| Class-D Kitaev chain | Exact flat periodic bulk spectrum at the solvable point | Pfaffian parity and one Majorana at each open end | Agreement with the nontrivial spin or Arf $\mathbb{Z}/2$ class under the low-energy interpretation [10, 8] |
| BDI stack | Exact free gap plus the Fidkowski-Kitaev gapped interaction for eight copies | Free integer, reduced to residue modulo eight with interactions | Agreement with the two-dimensional $\mathrm{Pin}^-$ Arf-Brown-Kervaire $\mathbb{Z}/8$ sector [7, 8] |
| Spin-one AKLT chain | Rigorous AKLT bulk-gap theorem | Nontrivial projective $SO(3)$ boundary representation | Agreement with the nontrivial $H^2(SO(3), U(1))$ sector [2, 3] |

Each row is microscopically realized in the stated one-dimensional setting. The last column records a limited comparison, not a spectrum equivalence.

### 12.2 Surjectivity and injectivity

Surjectivity of a realization map would require every abstract class to have a local Hamiltonian witness with a uniform gap and the correct response. Anomalies, finite-dimensional local Hilbert spaces, spatial symmetry, or unknown gap proofs may obstruct or delay such a result.

Injectivity would require two systems with the same abstract class to be connected by the chosen stabilized gapped path. A field-theory target may forget crystalline data, noninvertible excitations, boundary termination, or unstable finite-band structure. If the microscopic category retains any of these, injectivity can fail.

The BDI example shows that changing the domain can also merge classes. Once interactions are admitted, paths exist that were absent in the free category. Realization and comparison questions are meaningless until the domain and its morphisms are fixed.

## 13 Arrow-by-arrow theorem obligations

The five-stage program is best audited one arrow at a time.

| Arrow | Required theorem | Status | Main missing point |
|---|---|---|---|
| Interactions to Hamiltonian moduli | Uniform analytic bounds define a stack with controlled morphisms | PROPOSED | Higher descent and variable microscopic presentations |
| Hamiltonian moduli to gapped locus | Witness projection has pullback-stable propositional image | PROPOSED | Uniform thermodynamic witnesses and image descent |
| Gapped locus to stabilized phases | Controlled path localization and atomic stabilization commute with descent | PROPOSED | Global calculus of equivalences and coherences |
| Stabilized phases to invertible sector | Stacking descends as a symmetric monoidal structure and units form a Picard object | PROPOSED | Physical inverses and sheafwise invertibility |
| Picard object to spectrum | Internal grouplike recognition and spatial delooping | PROPOSED for connective spectrification, OPEN for spatial Omega-structure | Compatibility between spatial suspension and categorical delooping |
| Microscopic spectrum to K-theory | Natural invariant map preserving families and stacking | ESTABLISHED in controlled free sectors, OPEN generally | Interactions and information loss |

| Arrow | Required theorem | Status | Main missing point |
|---|---|---|---|
| Microscopic spectrum to EFT or bordism | Low-energy functor with stabilization and defect compatibility | OPEN generally | Existence, functoriality, faithfulness, and realization |
| Kubota spectrum to condensed spectrum | Comparison on a common smooth and condensed parameter domain | OPEN | Site change, gap conventions, and descent |

This table prevents progress on one arrow from being reported as completion of the sequence. Aoki's theorem, for example, advances an invariant bridge after a Banach algebra is given. It does not solve the moduli, gap, phase, or realization arrows.

# 14 Obstructions and failed shortcuts

Several tempting shortcuts are known to fail.

**Counterexample 14.1** (Pointwise gaps)**.** For every parameter $s$ and every finite volume, suppose the Hamiltonian has a positive gap $\gamma_{s,L}$. The infimum over $s$ and $L$ may still be zero. The data do not define a point of $\mathfrak{Gap}^{\mathrm{wit}}$ without one positive lower bound and a declared low-energy sector.

**Counterexample 14.2** (Qualitative image replaced by witnesses)**.** If a Hamiltonian admits two lower bounds $\gamma$ and $\gamma/2$, the witness object contains at least two points over the same Hamiltonian. Its projection cannot be the inclusion of a substack. The qualitative image must truncate the choice.

**Counterexample 14.3** (Homotopy groups without a map)**.** Two connective spectra can have isomorphic individual homotopy groups while their Postnikov data or multiplicative structures differ. A list of group agreements does not define an equivalence of spectra.

**Counterexample 14.4** (Finite-size numerical gap evidence)**.** A computation at lengths $L \leq 20$ can be consistent with a sequence of gaps that later tends to zero. It is evidence for a conjecture, not a thermodynamic witness.

**Counterexample 14.5** (Free invariant under interacting paths)**.** BDI phases of free indices zero and eight are distinct in the free path category and equivalent after the Fidkowski-Kitaev interaction is admitted. Thus the free K-class is not a complete invariant of the interacting phase category.

**Counterexample 14.6** (Bordism generator without lattice data)**.** A generator of a bordism or Anderson-dual group specifies an abstract field-theory deformation class. It does not specify local degrees of freedom, an interaction, a gap theorem, or a stacking inverse. Calling the class realized before those witnesses are supplied changes the meaning of realization.

# 15 A precise research agenda

The synthesis isolates five theorem packages. They can be pursued largely independently and then joined by explicit interfaces.

## 15.1 Package I: analytic family control

The first package should formulate a category of lattices and interactions on which a common locality norm implies family-uniform Lieb-Robinson bounds, thermodynamic dynamics, and restriction stability. It should include moving defects, disorder hulls, and variable local Hilbert spaces without confusing local continuity with uniform decay.

The main target is a base-change theorem: pullback of a controlled family preserves every constant used by the dynamics. A second target is finite descent for quantitative locality certificates. Infinite descent will need a bornological or boundedness condition that prevents constants from escaping.

## 15.2 Package II: gap certificates and openness

The second package should compare finite-volume and GNS witnesses under explicit boundary and sector hypotheses. Its output should be a stack of witnesses over the Hamiltonian moduli object, together with a theorem that the propositional image is pullback-stable.

A useful certificate has fields for the lower bound, volume threshold, ground-state sector, boundary convention, and theorem provider. This makes the gap claim transportable without pretending that the provider is unique.

## 15.3 Package III: phase localization

The third package should define controlled gapped paths, quasi-local automorphisms, finite-depth circuits, lattice changes, and stabilization in one infinity-categorical model. Comparison functors between these notions should be theorems, not definitional equalities.

The practical question is whether local phase equivalences glue. On an overlap, two spectral-flow providers may differ by a phase automorphism. That difference belongs in the higher groupoid and cannot be erased during descent.

## 15.4 Package IV: stacking and spectrification

The fourth package should prove that stacking preserves locality and gap witnesses with controlled constants, descends through localization, and is coherently symmetric monoidal. It should then distinguish actual physical inverses from formal Grothendieck inverses.

Connective spectrification is only part of the job. To compare spatial dimensions, one needs a theorem that adding a lattice direction implements the loop or suspension maps of an Omega-spectrum. Kubota's construction is the clearest established benchmark for this step.

### 15.5 Package V: comparison and realization

The fifth package should build natural transformations on carefully chosen overlaps:

1. from free condensed Hamiltonian families to operator K-theory;
2. from Kubota's smooth microscopic families to condensed tests;
3. from controlled microscopic invertible phases to reflection-positive effective field theories;
4. from a generalized cohomology invariant to relative transition classes;
5. from abstract classes back to explicit microscopic witnesses where possible.

Each map should list the information it forgets. A realization theorem should state its locality, symmetry, dimensional, and boundary restrictions.

## 16 Formal and executable representations

The repository contains a Lean interface library for the shared vocabulary of the project. It records claim statuses, providers, locality and gap witness interfaces, objectwise C*-conditions, stacking, phase equivalence, and comparison records. It does not assert the deep analytic theorems as axioms. This synthesis does not modify that library.

The Haskell program accompanying this paper has a narrower purpose. It checks the architecture of a proposed synthesis record. Its data types distinguish the witness fibration from the propositional image and distinguish an invariant map from an equivalence theorem. Its tests reject:

1. a phase sequence that skips a stage or reverses its order;
2. a qualitative gapped locus represented by witness data;
3. an equivalence claim without a named comparison theorem;
4. a physical relative charge without a microscopic comparison;
5. an untwisted Thom target without an orientation;
6. a spacetime degree that fails $n = d + 1$.

The executable is not a proof assistant. It verifies a finite status and dependency contract. Analytic results remain cited theorem providers.

## 17 Claim ledger

| Statement | Status | Basis and boundary |
|---|---|---|
| Uniform Lieb-Robinson bounds and thermodynamic dynamics | Established | Common interaction norm, bounded geometry, and uniform tail hypotheses |
| C*-norm and positivity for $C(S, \mathcal{A})$ | Established | A fixed C*-algebra $\mathcal{A}$ and compact Hausdorff parameter space |
| Finite-cover C*-descent | Established | Finite jointly surjective profinite cover and continuous gluing |
| Bare algebraic data determine a unique physical C*-norm | Obstructed | Distinct C*-completions provide counterexamples |
| Uniform gap stability and spectral flow | Established | Controlled paths satisfying the established locality, regularity, and uniform-gap hypotheses |
| Pointwise gaps imply a family-uniform thermodynamic gap | Obstructed | Positive numbers can have zero infimum; moving defects give physical models |
| Witnessed gap object projects as a fibration | Proposed | Explicit dependent-pair construction with many witnesses per Hamiltonian |
| Propositional gapped substack | Proposed | Requires pullback-stable image factorization and descent of common witnesses |
| Stabilized phase infinity-groupoid | Proposed | Requires controlled localization, ancilla stabilization, and higher descent |
| Invertible condensed phase spectrum | Proposed | Conditional internal spectrification of the Picard phase object |
| Spatial Omega-spectrum structure for the condensed construction | Open | Needs a suspension or lattice-direction delooping theorem |
| Controlled free flattening to operator K-theory | Established | Bounded self-adjoint family with a common Fermi gap |
| Aoki solidification comparison | Established | Algebraic to topological K-theory for real or complex Banach algebras, with connective and Bott qualifications |
| Free K-theory classifies all interacting phases | Obstructed | BDI interaction reduction supplies a counterexample |
| Kubota microscopic Omega-spectrum | Established | Invertible gapped spin systems with the hypotheses of Kubota's construction |
| Freed-Hopkins field-theory classification | Established | Reflection-positive invertible field theories under the stated tangential and field-theoretic hypotheses |

| Statement | Status | Basis and boundary |
|---|---|---|
| Kubota spectrum equals the condensed phase spectrum | Open | No comparison equivalence has been constructed |
| Microscopic spectrum equals the Freed-Hopkins target | Open | Low-energy functor, family compatibility, and realization are missing generally |
| Kitaev, BDI, and AKLT witness rows | Established | Explicit interactions, symmetries, gap calculations or theorems, and microscopic invariants |
| Every bordism class is physically realizable | Open | No general surjectivity theorem is asserted |
| Relative connecting class $\partial\nu$ | Established | Long exact sequence of the pair $(B, B \setminus \Sigma)$ |
| Thom localization to $E^{q+1-c}(\Sigma)$ | Established | Excision, smooth closed locus, and $E$-oriented rank-$c$ normal bundle |
| Every relative class is a physical transition charge | Obstructed | A microscopic invariant and natural comparison map are additional required data |

# 18 Discussion

The condensed perspective earns its keep by forcing parameter dependence to be part of the object. Disorder, inverse limits, and higher families then belong to the same moduli problem as ordinary paths. It also forces descent questions into the open. A family assembled from local pieces needs common analytic constants, not just compatible point values.

The separation of witnessed and qualitative gaps is indispensable. A gap proof contains a lower bound, a ground-state convention, and often a theorem-specific presentation. Those choices matter during verification and are irrelevant to the yes-or-no gapped locus. The witness fibration retains them; the propositional image forgets them. Conflating the two damages both the mathematics and the software model.

Stable-homotopy objects in this subject also come from different physical categories. Operator K-theory is an invariant of a specified operator algebra. Kubota's spectrum is built from microscopic gapped spin systems and smooth parameter families. Freed-Hopkins classify invertible field theories under reflection positivity and tangential hypotheses. The proposed condensed spectrum is intended to organize condensed parameter families of microscopic phases. Similar grading patterns make comparison reasonable. They do not make comparison automatic.

Transition charges expose the same issue in a smaller calculation. The expression $E^{q+1}(B, B \setminus \Sigma)$ is exact algebraic topology. Its physical content begins only after a microscopic phase invariant lands in $E^q(U)$. Excision localizes the obstruction, and a Thom isomorphism moves it onto the critical locus under an orientation hypothesis. None of these steps can replace the microscopic comparison map.

The program is therefore neither complete nor empty. Its analytic foundations include locality bounds, C*-control, gap stability in established settings, functional calculus, and explicit low-dimensional realizations. Its categorical assembly is a precise proposal with identifiable theorem obligations. That is enough to support a focused research program without claiming a classification that has not been built.

# 19 Conclusion

The five-stage sequence can be made mathematically coherent if every stage keeps its hypotheses and every arrow keeps its own proof obligation:

$$\boxed{\begin{array}{c}\text{local quantum interactions}\\ \downarrow \\ \text{condensed moduli stack of Hamiltonians}\\ \downarrow \\ \text{uniformly gapped propositional substack}\\ \downarrow \\ \text{stabilized phase } \infty\text{-groupoid}\\ \downarrow \\ \text{invertible condensed phase spectrum.}\end{array}}$$

Locality comes from uniform interaction estimates. Positivity and norms come from a declared C*-algebraic setting. A thermodynamic gap comes from a quantitative witness. The qualitative gapped substack is the propositional image of the witness fibration. Phases arise only after controlled path localization and stabilization. The spectrum arises only after coherent stacking, restriction to physical units, and internal spectrification.

Under these conditions, $\pi_0$ records invertible stabilized phases, $\pi_1$ records loops that become pumps when analytic transport is supplied, and higher homotopy records higher families that become defects only through a family-to-defect theorem. The gapless locus is pulled back from the propositional gapped subobject. A relative class $\partial\nu \in E^{q+1}(B, B \setminus \Sigma)$ measures failure of an invariant to extend, while a physical charge also needs a microscopic comparison.

The proposed spectrum must remain distinct from operator K-theory, Kubota's microscopic Omega-spectrum, and the Freed-Hopkins field-theory target until comparison theorems say otherwise. Explicit models can test those maps. They cannot be replaced by matching tables of groups. The next advances should therefore prove arrows, not rename their endpoints.

# A The five stage data contract

This appendix records a compact contract suitable for formalization.

## A.1 Local interaction record

A local interaction family record contains

$$(\Gamma, d, \mathcal{A}_{\mathrm{loc}}, \Phi, F, C_F, T_F, G, \alpha), \tag{46}$$

where $C_F$ is a common interaction-norm bound and $T_F$ is a common tail certificate. Restriction may change the parameter object but may not weaken the declared bound silently.

## A.2 Hamiltonian moduli record

A Hamiltonian moduli record adds a test site, object and morphism categories, restriction functors, descent data, and size conventions. The object record alone defines a sheaf of data at best. Higher moduli require mapping anima and coherent descent.

## A.3 Gap record

A gap witness contains

$$(\gamma, L_0, \mathrm{sector}, \mathrm{boundary}, \mathrm{provider}) \tag{47}$$

or its GNS analogue. The qualitative flag is the truncation of the existence of such a record. The flag never substitutes for the provider during an analytic proof.

## A.4 Phase record

A phase record lists its allowed paths, local circuits, quasi-local automorphisms, lattice changes, ancillas, and symmetry rules. Two papers that use different records may obtain different phase sets without contradiction.

## A.5 Spectrum record

A spectrum record lists the symmetric monoidal operation, unit, physical inverse witness, $E_\infty$ coherences, test-object descent, and spatial suspension maps. A sequence of abelian groups is not a substitute for this record.

# B Comparison theorem checklist

Before identifying two phase constructions, a comparison should answer all of the following questions.

| Field | Required answer |
|---|---|
| Source objects | Local interactions, fields, algebras, or phase objects in the source |
| Source morphisms | Gapped paths, quasi-local maps, field-theory deformations, or stable maps |

| Field | Required answer |
|---|---|
| Target objects | Exact target category or spectrum, including grading conventions |
| Parameter site | Smooth manifolds, compact spaces, profinite sets, or condensed objects |
| Analytic domain | Locality norm, gap, regularity, symmetry, and boundary hypotheses |
| Formula | Construction of the map on objects and parameter families |
| Naturality | Compatibility with pullback and restriction |
| Homotopy invariance | Proof that source equivalences map to target equivalences |
| Stacking | Symmetric monoidal compatibility and behavior of units |
| Suspension | Compatibility with dimension shifts and spectrum structure |
| Defects | Treatment of boundaries, higher morphisms, and anomaly inflow |
| Faithfulness | Which distinct microscopic phases can the target merge? |
| Fullness | Which target morphisms lift to microscopic morphisms? |
| Surjectivity | Which target objects have explicit microscopic realizations? |
| Lost data | Energy scales, dispersion, correlation lengths, boundaries, or interactions forgotten by the map |
| Inverse | Explicit construction and proof, if an equivalence is claimed |

# C Relative charge bookkeeping

Let $j : U \hookrightarrow B$. The relevant part of the exact sequence is

$$E^q(B) \xrightarrow{j^*} E^q(U) \xrightarrow{\partial} E^{q+1}(B, U) \longrightarrow E^{q+1}(B). \tag{48}$$

The checks are:

1. the connecting map raises degree by one;
2. exactness gives $\partial j^* = 0$;
3. excision moves the pair to a neighborhood of $\Sigma$;
4. a rank-$c$ $E$-orientation shifts the localized degree down by $c$;
5. no orientation means a twisted Thom target;
6. a physical name requires a microscopic comparison map.

For a codimension-one transition locus, the Thom shift returns degree $q$. For an isolated codimension-three degeneracy, it gives degree $q-2$ on the point, equivalently a linking class on an $S^2$ in the normal coordinates.

# D Status-sensitive glossary

| Term | Meaning in this paper |
|---|---|
| Local family | Parameter family with continuous local coefficients and common decay bounds |
| Condensed moduli stack | Proposed anima-valued sheaf of Hamiltonian objects and controlled equivalences |
| Gap witness | Quantitative finite-volume or GNS certificate with a positive common lower bound |
| Witness fibration | Dependent object of Hamiltonians together with chosen certificates |
| Propositional image | Qualitative subobject asserting only that some witness exists |
| Thermodynamic gap | Positive bulk-gap lower bound uniform in the volume convention |
| Phase | Equivalence class in the chosen localized and stabilized microscopic category |
| Invertible phase | Phase with a represented stacking inverse |
| Condensed spectrum | Proposed internal connective spectrum built from the Picard phase object |
| Operator K-class | Stable invariant of a specified operator algebra, not a microscopic reconstruction |
| Kubota spectrum | Established microscopic Omega-spectrum in Kubota's smooth almost-local setting |
| Freed-Hopkins target | Stable-homotopy classification of reflection-positive invertible field theories under its hypotheses |
| Gapless locus | Complement of the pulled-back propositional gapped locus when that complement is represented |
| Relative class | Connecting image $\partial\nu$ in the cohomology of a pair |
| Physical transition charge | Relative class equipped with a microscopic invariant and natural comparison map |
| Realized class | Abstract label with a complete microscopic witness in the stated rules |

# E Source papers and division of labor

The five companion papers supply separate theorem packages.

1. *Locality and Lieb-Robinson Bounds for Condensed Families* isolates uniform interaction norms, family continuity, thermodynamic dynamics, and the moving-defect counterexample.

2. *Condensed Observable Algebras* treats C*-norms, positivity, finite descent, state spaces, crossed products, and Aoki's solidification bridge.

3. *Uniform Thermodynamic Spectral Gaps in Condensed Families* separates finite-volume and GNS witnesses and distinguishes the witness fibration from its propositional image.

4. *From Microscopic Lattice Hamiltonians to Effective Field Theories* proves the controlled free functional-calculus map, works out the SSH transition, and documents information loss and the BDI reduction.

5. *Physical Realizability of Invertible Phase Classes* compares the Freed-Hopkins, Kubota, and proposed condensed targets, verifies three low-dimensional witnesses, and states the exact relative-charge hypotheses.

The synthesis uses their accepted claims without promoting any companion proposal to an established theorem.

# References


[1] K. Aoki, *(Semi)topological K-theory via solidification*, arXiv:2409.01462, 2024.

[2] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, *Rigorous results on valence-bond ground states in antiferromagnets*, Physical Review Letters **59** (1987), 799 to 802, https://doi.org/10.1103/PhysRevLett.59.799.

[3] S. Bachmann and B. Nachtergaele, *On gapped phases with a continuous symmetry and boundary operators*, Journal of Statistical Physics **154** (2014), 91 to 112, https://arxiv.org/abs/1307.0716.

[4] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, *Automorphic equivalence within gapped phases of quantum lattice systems*, Communications in Mathematical Physics **309** (2012), 835 to 871, https://arxiv.org/abs/1102.0842.

[5] B. Blackadar, *K-Theory for Operator Algebras*, second edition, Cambridge University Press, 1998.

[6] P. Scholze, *Lectures on Condensed Mathematics*, arXiv:2605.03658, 2026.

[7] L. Fidkowski and A. Kitaev, *The effects of interactions on the topological classification of free fermion systems*, Physical Review B **81** (2010), 134509, https://doi.org/10.1103/PhysRevB.81.134509.

[8] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, Geometry and Topology **25** (2021), 1165 to 1330, https://doi.org/10.2140/gt.2021.25.1165.

[9] M. B. Hastings and T. Koma, *Spectral gap and exponential decay of correlations*, Communications in Mathematical Physics **265** (2006), 781 to 804, https://doi.org/10.1007/s00220-006-0030-4.

[10] A. Y. Kitaev, *Unpaired Majorana fermions in quantum wires*, Physics-Uspekhi **44** (2001), 131 to 136, https://arxiv.org/abs/cond-mat/0010440.

[11] Y. Kubota, *Stable homotopy theory of invertible gapped quantum spin systems I: Kitaev's Omega-spectrum*, arXiv:2503.12618, 2025.

[12] E. H. Lieb and D. W. Robinson, *The finite group velocity of quantum spin systems*, Communications in Mathematical Physics **28** (1972), 251 to 257.

[13] B. Nachtergaele and R. Sims, *Lieb-Robinson bounds in quantum many-body physics*, Contemporary Mathematics **529** (2010), 141 to 176, https://arxiv.org/abs/1004.2086.

[14] E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer, 2016.

[15] J. C. Y. Teo and C. L. Kane, *Topological defects and gapless modes in insulators and superconductors*, Physical Review B **82** (2010), 115120, https://doi.org/10.1103/PhysRevB.82.115120.

[16] G. C. Thiang, *On the K-theoretic classification of topological phases of matter*, Annales Henri Poincare **17** (2016), 757 to 794, https://doi.org/10.1007/s00023-015-0418-9.