Uniform Locality for Condensed Families of Quantum Lattice Interactions

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Abstract

We isolate the analytic input needed to place parameterized quantum lattice interactions in a condensed mathematical setting. The decisive condition is not profiniteness of the parameter object. It is a bound on an interaction F-norm that is uniform in the parameter. Under the usual summability and convolution conditions on F, such a bound gives finite-volume Lieb-Robinson estimates with constants independent of both the parameter and the volume. It also gives thermodynamic dynamics, uniformly on compact time intervals. A separate local F-continuity condition yields point-norm continuity of the infinite-volume dynamics on local observables. We package these hypotheses into a functor on profinite sets and prove its elementary finite-cover descent property. This packaging is categorical. It supplies none of the propagation estimates by itself.

Two counterexamples mark the boundary of the result. First, compact parameterization and pointwise locality do not imply a uniform interaction bound. Second, even local parameter continuity together with a uniform locality bound and a positive gap in every fiber does not imply a gap uniform in the family. The latter failure is exhibited by an on-site defect whose strength tends to zero while its position escapes to infinity. We state all results with exact hypotheses, distinguish established analytic theorems from the proposed condensed organization, and include finite executable checks in Haskell. The outcome is a precise first arrow in the program from local quantum interactions to a condensed moduli stack of Hamiltonians.

Contents

1 Introduction

Topological phases are usually organized by paths of gapped Hamiltonians, stable equivalence, operator algebras, or effective field theories. Each of these viewpoints begins with a more elementary question. Which families of lattice interactions define controlled dynamics in the first place? A classification of phases cannot be more reliable than its locality hypotheses.

Condensed mathematics suggests testing a moduli object on profinite sets. If S is profinite, an S-point should describe an S-parameterized family of interactions. This is attractive for disorder spaces such as $D^{\mathbb{Z}^d}$ with D finite. It is also potentially misleading. The sheaf condition glues compatible data over a finite cover. It does not bound an infinite sum of interaction terms, prove a convolution estimate, or construct thermodynamic time evolution.

The purpose of this paper is to separate these jobs. The analytic layer is provided by interaction Banach spaces and Lieb-Robinson theory. The parameter layer records local continuity and a common norm bound. The condensed layer records functoriality in a profinite test object and descent for finite jointly surjective covers.

Our main theorem is a uniform-family corollary of the standard Lieb-Robinson and thermodynamic-limit theorems. Let (Γ, d) be a countable metric space, let F satisfy uniform summability and a convolution condition, and let $(\Phi_s)_{s\in S}$ be a family of interactions such that

$$\sup_{s\in S} \|\Phi_s\|_F \leq M < \infty. \tag{1}$$

Then the propagation constants depend on F and M, not on s or the finite volume. The corresponding thermodynamic dynamics exist with uniform finite-volume convergence on compact time intervals. If the family is locally continuous in the sense defined below, then $s \mapsto \tau_t^s(A)$ is norm-continuous for each local observable A.

The compactness of S is useful only after a topology has been specified. For example, continuity of $S \to \mathcal{B}_F$ implies equation (1) when S is compact. Product-topology disorder families are generally not continuous in this global Banach norm. They may still satisfy the common bound directly and may be continuous on every local restriction. Keeping these two requirements separate is essential.

The paper uses four claim labels.

- Established marks a cited theorem or a direct corollary with proof.

- PROPOSED marks the categorical organization introduced here.
- OBSTRUCTED marks a statement ruled out by a counterexample.
- OPEN marks a comparison or extension not proved here.

The central claim of this paper is ESTABLISHED. The condensed moduli functor is PROPOSED. The assertion that profiniteness by itself produces uniform locality is OBSTRUCTED.

1.1 Main contributions

The contributions are deliberately narrow.

1. We fix a single interaction norm and state the corresponding Lieb-Robinson estimate with its constants.
2. We prove a uniform-family theorem for finite-volume propagation and thermodynamic dynamics.
3. We formulate a local F-topology suitable for product-type disorder families and prove parameter continuity of dynamics under a common tail bound.
4. We define a profinite functor of uniformly local interactions and prove finite-cover descent.
5. We give counterexamples separating pointwise locality, uniform locality, local continuity, and uniform gappedness.
6. We provide a finite Haskell companion that computes F-function diagnostics, Lieb-Robinson envelopes, and finite witnesses for the logical difference between pointwise and uniform conditions.

We do not prove a spectral gap, construct a phase spectrum, or identify a microscopic lattice category with an effective field theory. Those problems begin after the analytic foundation established here.

1.2 Relation to the literature

The norm and propagation estimates follow the framework developed by Nachtergaele and Sims and its detailed treatment by Nachtergaele, Sims, and Young [4, 7]. The thermodynamic-limit argument uses a uniform Cauchy estimate on local observables. Parameter dependence is an application of continuity and local F-convergence estimates in [7]. Spectral flow is discussed only to identify which output of this paper later phase-classification work needs [6]. No theorem from condensed mathematics is used to prove an analytic estimate.

2 Quantum lattice systems and interaction norms

2.1 The quasi-local algebra

Definition 2.1 (Quantum lattice system)**.** Let (Γ, d) be a countable metric space. To every $x \in \Gamma$ assign a finite-dimensional complex Hilbert space $\mathcal{H}_x$. For a finite set $\Lambda \Subset \Gamma$, define

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x, \qquad \mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda). \tag{2}$$

If $\Lambda_0 \subseteq \Lambda$, identify $A \in \mathcal{A}_{\Lambda_0}$ with $A \otimes 1_{\Lambda\setminus\Lambda_0}$. The local algebra and quasi-local algebra are

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\Subset\Gamma} \mathcal{A}_\Lambda, \qquad \mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|}. \tag{3}$$

Finite-dimensional on-site spaces keep the presentation focused. The Lieb-Robinson estimate permits certain unbounded on-site Hamiltonians after passing to an interaction picture, but unbounded interaction terms require additional hypotheses and are outside our scope.

Definition 2.2 (Interaction)**.** An interaction is a map

$$\Phi : \mathcal{P}_0(\Gamma) \longrightarrow \mathcal{A}_\Gamma^{\mathrm{loc}} \tag{4}$$

such that $\Phi(Z) = \Phi(Z)^* \in \mathcal{A}_Z$ for every finite Z. Its finite-volume Hamiltonian is

$$H_\Lambda^\Phi = \sum_{Z\subseteq\Lambda} \Phi(Z). \tag{5}$$

The associated finite-volume Heisenberg dynamics is

$$\tau_t^{\Lambda,\Phi}(A) = e^{itH_\Lambda^\Phi} A e^{-itH_\Lambda^\Phi}. \tag{6}$$

All finite-volume expressions are well defined. The substantive issue is to control them independently of Λ.

2.2 F-functions

Definition 2.3 (F-function)**.** A non-increasing function $F : [0,\infty) \to (0,\infty)$ is an F-function on (Γ, d) if

$$\|F\| := \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) < \infty \tag{7}$$

and

$$C_F := \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) < \infty. \tag{8}$$

The first condition says that tails are uniformly summable. The second says that one convolution does not degrade the spatial profile beyond a constant. Repeated commutator estimates then produce powers of C_F without changing the form of the decay function.

Example 2.4 (Polynomial and exponential weights)**.** On $\mathbb{Z}^\nu$ with the ℓ^1 metric, for every $\epsilon > 0$,

$$F(r) = \frac{1}{(1+r)^{\nu+\epsilon}} \tag{9}$$

is an F-function. If g is non-decreasing and subadditive, then $F_g(r) = e^{-g(r)}F(r)$ is again an F-function. In particular,

$$F_a(r) = e^{-ar}F(r), \qquad a > 0, \tag{10}$$

produces the familiar exponential light-cone form.

Definition 2.5 (Interaction F-norm)**.** For an interaction Φ, set

$$\|\Phi\|_F = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{Z\in\Gamma\\ x,y\in Z}} \|\Phi(Z)\|. \tag{11}$$

Let $\mathcal{B}_F$ be the vector space of interactions with finite F-norm.

The $x = y$ part controls the total interaction incident on one site. The $x \neq y$ part controls simultaneous incidence at separated sites. This is stronger than asking each individual term to decay with its diameter.

Lemma 2.6 (Pair bound)**.** *If $\Phi \in \mathcal{B}_F$, then for all $x, y \in \Gamma$,*

$$\sum_{Z\ni x,y} \|\Phi(Z)\| \le \|\Phi\|_F F(d(x,y)). \tag{12}$$

Proof. This is the defining inequality obtained by multiplying equation (11) by $F(d(x,y))$. □

Lemma 2.7 (Uniform tail)**.** *Define*

$$T_F(R) = \sup_{x\in\Gamma} \sum_{\substack{y\in\Gamma\\ d(x,y)\ge R}} F(d(x,y)). \tag{13}$$

Then $T_F(R) \to 0$ as $R \to \infty$ whenever the uniform summability in equation (7) is accompanied by the usual uniform tail property. On a uniformly locally finite lattice and for the standard polynomial or weighted polynomial examples, this property holds.

Proof. For the standard lattice examples, compare the sum with the corresponding radial series. More generally, the tail property is recorded separately because boundedness of a family of summable functions does not force uniform integrability without a geometric hypothesis. All later uniform convergence statements assume $T_F(R) \to 0$. □

Remark 2.8. This explicit tail assumption is useful in a family theorem. It prevents a silent exchange of a supremum over base points with a limit in the radius. For $\mathbb{Z}^\nu$ and the F-functions used in applications, no extra work is needed.

3 Parameterized interaction families

3.1 Uniform locality and local continuity

Let S be a compact Hausdorff space. The intended applications include profinite sets, compact coupling spaces, and compact disorder hulls.

Definition 3.1 (Uniformly F-local family)**.** A family $\Phi = (\Phi_s)_{s\in S}$ is uniformly F-local if

$$M_F(\Phi) := \sup_{s\in S} \|\Phi_s\|_F < \infty. \tag{14}$$

This is a property of the whole family. It is not the statement that each $\|\Phi_s\|_F$ is finite.

Definition 3.2 (Local F-continuity)**.** A uniformly F-local family is locally F-continuous if, for every finite $\Lambda \Subset \Gamma$, the map

$$S \longrightarrow \mathcal{B}_F, \qquad s \longmapsto \Phi_s|_\Lambda \tag{15}$$

is continuous, where $\Phi_s|_\Lambda$ retains only terms supported inside Λ. Equivalently, for every net $s_i \to s$ and finite Λ,

$$\|(\Phi_{s_i} - \Phi_s)|_\Lambda\|_F \longrightarrow 0. \tag{16}$$

Since a finite Λ has only finitely many subsets, local F-continuity is equivalent to norm continuity of every coefficient map $s \mapsto \Phi_s(Z)$ for finite $Z \Subset \Gamma$. This coordinate description is useful, but the restricted F-norm is better suited to the estimates below.

Terms crossing the boundary of Λ are not ignored analytically. Their effect is controlled uniformly by the F-tail after Λ is enlarged. This is why local continuity must be paired with equation (14).

Proposition 3.3 (A sufficient global condition)**.** *If S is compact and $s \mapsto \Phi_s$ is continuous as a map into the Banach space $\mathcal{B}_F$, then the family is uniformly F-local and locally F-continuous.*

Proof. The norm is continuous, so its image on the compact set S is bounded. Restriction to a finite volume is a continuous contraction in the relevant interaction norm. $\square$

Remark 3.4. The converse is false and should not be desired. In a product disorder space, two configurations that agree near the origin can disagree at every distant scale. A covariant interaction may therefore be close on every fixed local region without being close in a norm that takes a supremum over all lattice sites.

3.2 Finite-volume parameter continuity

Lemma 3.5. *If Φ is locally F-continuous, then for fixed finite Λ the map*

$$(s, t, A) \longmapsto \tau_t^{\Lambda,\Phi_s}(A) \tag{17}$$

is norm-continuous on $S \times \mathbb{R} \times \mathcal{A}_\Lambda$.

Proof. The finite sum $H_\Lambda^{\Phi_s}$ is norm-continuous in s. The exponential map is norm-continuous on finite-dimensional matrix algebras, and conjugation is jointly continuous. $\square$

This elementary lemma is not enough for the thermodynamic limit. The number of interaction terms grows with the volume. Uniform locality is what makes the finite statement stable as Λ tends to Γ.

4 Uniform Lieb-Robinson estimates

4.1 The finite-volume bound

For finite $X \subseteq \Gamma$, let $\partial_\Phi X$ be the set of sites in X belonging to a nonzero interaction term, meaning a set Z with $\Phi(Z) \neq 0$, that also meets $\Gamma \setminus X$. For disjoint finite X, Y, define

$$D_{F,\Phi}(X,Y) = \min\left\{\sum_{x\in X}\sum_{y\in\partial_\Phi Y} F(d(x,y)), \sum_{x\in\partial_\Phi X}\sum_{y\in Y} F(d(x,y))\right\}. \tag{18}$$

We also use the coarser parameter-independent quantity

$$\overline{D}_F(X,Y) = \sum_{x\in X}\sum_{y\in Y} F(d(x,y)). \tag{19}$$

Clearly $D_{F,\Phi}(X,Y) \leq \overline{D}_F(X,Y)$.

Theorem 4.1 (Lieb-Robinson bound)**.** *Let $\Phi \in \mathcal{B}_F$. Let $X, Y \Subset \Gamma$ be disjoint, $\Lambda \supseteq X \cup Y$, $A \in \mathcal{A}_X$, and $B \in \mathcal{A}_Y$. Then*

$$\|[\tau_t^{\Lambda,\Phi}(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}\left(e^{2C_F\|\Phi\|_F|t|} - 1\right) D_{F,\Phi}(X,Y). \tag{20}$$

In particular, the same estimate holds with $D_{F,\Phi}$ replaced by $\overline{D}_F$.

Proof sketch with the analytic dependence displayed. For $B \in \mathcal{A}_Y$, consider the commutator map $A \mapsto [\tau_t^{\Lambda,\Phi}(A), B]$. Differentiate after removing the internal dynamics on X. Only interaction terms crossing the current support boundary enter the resulting integral inequality. Iterating that inequality produces paths of overlapping interaction supports. The nth-order term is bounded by

$$2\|A\|\|B\|\frac{(2|t|\|\Phi\|_F)^n}{n!}C_F^{n-1}D_{F,\Phi}(X,Y). \tag{21}$$

Summing over $n \geq 1$ gives equation (20). The convolution estimate equation (8) is exactly what turns the sum over intermediate lattice sites into $C_F^{n-1}F(d(x,y))$. A complete proof, including time-dependent interactions and unbounded on-site terms, is Theorem 3.1 of [7]. $\square$

Theorem 4.2 (Uniform family Lieb-Robinson estimate)**.** *Let $(\Phi_s)_{s\in S}$ be uniformly F-local with $M = M_F(\Phi)$. Under the hypotheses of theorem 4.1,*

$$\|[\tau_t^{\Lambda,\Phi_s}(A), B]\| \le \frac{2\|A\|\|B\|}{C_F}\left(e^{2C_F M|t|} - 1\right)\overline{D}_F(X,Y) \tag{22}$$

for every $s \in S$ and every finite $\Lambda \supseteq X \cup Y$.

Proof. Apply theorem 4.1 to each Φ_s, use $\|\Phi_s\|_F \le M$, and replace the interaction-dependent boundary factor by equation (19). No compactness or topology on S is used. □

Remark 4.3 (Status). The single-interaction estimate is ESTABLISHED. The family statement is a direct corollary, also ESTABLISHED. The uniformity comes from the common norm bound, not from the word "family" and not from profiniteness.

4.2 Exponential light-cone form

Suppose $F_a(r) = e^{-ar}F(r)$ and the family is uniformly F_a-local. Define

$$v_a = \frac{2C_{F_a}M_{F_a}(\Phi)}{a}. \tag{23}$$

Corollary 4.4. *For disjoint X, Y,*

$$\begin{aligned}\|[\tau_t^{\Lambda,\Phi_s}(A), B]\| \le 2\|A\|\|B\|\|F\|C_{F_a}^{-1}\min\{|X|,|Y|\}\\ \times \exp\big(a(v_a|t| - d(X,Y))\big),\end{aligned} \tag{24}$$

uniformly in s and Λ.

Proof. Subadditivity of distance extracts $e^{-ad(X,Y)}$ from the spatial sum. Uniform summability bounds the remaining sum by $\|F\|\min\{|X|,|Y|\}$. Rewriting the time exponential with the definition of v_a gives equation (24). □

The number v_a is a velocity bound, not necessarily the optimal physical velocity. The theorem controls a light-cone envelope and makes no assertion that a signal saturates it.

5 Thermodynamic dynamics

5.1 Existence and uniform convergence

Let $(\Lambda_n)_{n\ge1}$ be increasing and exhaustive. For a fixed local observable $A \in \mathcal{A}_X$, choose n_0 with $X \subseteq \Lambda_{n_0}$.

Theorem 5.1 (Uniform thermodynamic dynamics)**.** *Let $(\Phi_s)_{s\in S}$ be uniformly F-local, and assume the uniform F-tail tends to zero. For every $s \in S$, $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$, and $t \in \mathbb{R}$, the limit*

$$\tau_t^s(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n,\Phi_s}(A) \tag{25}$$

exists in norm. It is independent of the exhaustion. Convergence is uniform in s and in t on compact intervals. Each $(\tau_t^s)_{t\in\mathbb{R}}$ extends uniquely to a strongly continuous group of $*$-automorphisms of $\mathcal{A}_\Gamma$.

Proof. For $n \geq m \geq n_0$, the standard comparison estimate for two finite-volume dynamics gives

$$\begin{aligned} &\|\tau_t^{\Lambda_n,\Phi_s}(A) - \tau_t^{\Lambda_m,\Phi_s}(A)\| \\ &\qquad \leq \frac{2\|A\|}{C_F} e^{2C_F M|t|} C_F M|t| \sum_{x\in X} \sum_{y\in\Lambda_n\setminus\Lambda_m} F(d(x,y)). \end{aligned} \tag{26}$$

The right side is independent of s. It tends to zero as $m, n \to \infty$, uniformly for t in a compact interval. Thus the finite-volume dynamics form a uniform Cauchy net on local observables. The group, involution, norm, and multiplicative properties pass to the limit. Exhaustion independence follows by interlacing two exhaustions. Strong continuity holds first on the dense local algebra using a uniform small-time estimate and then on all of $\mathcal{A}_\Gamma$ by isometry. This is the family-uniform version of Theorem 3.5 in [7]. □

Corollary 5.2 (Infinite-volume Lieb-Robinson bound)**.** *The estimate equation (22) holds with τ_t^{Λ,Φ_s} replaced by τ_t^s.*

Proof. Take the norm limit in the finite-volume commutator. The bound is independent of the volume. □

5.2 Continuity in the family parameter

Theorem 5.3 (Point-norm parameter continuity)**.** *Assume the hypotheses of theorem 5.1 and local F-continuity. For every local A and fixed t,*

$$S \longrightarrow \mathcal{A}_\Gamma, \qquad s \longmapsto \tau_t^s(A) \tag{27}$$

is norm-continuous. Continuity is uniform in t on compact intervals.

Proof. Fix $s \in S$, a net $s_i \to s$, a local observable $A \in \mathcal{A}_X$, a compact time interval $[-T, T]$, and $\epsilon > 0$. By equation (26), choose a finite Λ containing X such that

$$\sup_{r\in S,\, |t|\leq T} \|\tau_t^r(A) - \tau_t^{\Lambda,\Phi_r}(A)\| < \epsilon/3. \tag{28}$$

For this fixed Λ, local F-continuity implies $H_\Lambda^{\Phi_{s_i}} \to H_\Lambda^{\Phi_s}$ in norm. Finite-volume parameter continuity then gives, eventually and uniformly for $|t| \leq T$,

$$\|\tau_t^{\Lambda,\Phi_{s_i}}(A) - \tau_t^{\Lambda,\Phi_s}(A)\| < \epsilon/3. \tag{29}$$

The triangle inequality proves the claim. This is also a specialization of the local F-convergence theorem in [7]. □

Remark 5.4. Global $\mathcal{B}_F$-continuity gives a direct quantitative estimate for the difference of two dynamics. The local argument above is weaker and better adapted to disorder. It works because a local observable sees a large but finite region up to a uniformly controlled tail.

6 Profinite parameters and condensed organization

6.1 Why profinite spaces occur

Let D be finite. The product

$$\Omega = D^{\mathbb{Z}^d} \tag{30}$$

is compact, Hausdorff, and totally disconnected, hence profinite. Its clopen cylinder sets record finite patches of a disorder configuration. A local rule assigning interaction terms from finite patches is continuous in the product topology. Uniform boundedness of the allowed coupling values can give a common F-norm estimate, but this is an additional calculation.

Profinite sets also arise as inverse limits of finite-resolution parameter spaces. The useful fact is that continuous maps out of a profinite set can be approximated, in suitable uniform targets, by locally constant maps. For interaction families, however, approximation must respect the uniform F-bound. Finite-resolution language does not excuse a missing tail estimate.

6.2 A functor of controlled interactions

Fix (Γ, d), the on-site algebras, and an F-function. For $M > 0$ and a profinite set S, define

$$\mathfrak{Ham}_F^M(S) = \left\{ (\Phi_s)_{s\in S} : \begin{array}{l} \|\Phi_s\|_F \le M \text{ for every } s, \\ s \mapsto \Phi_s \text{ is locally } F\text{-continuous} \end{array} \right\}. \tag{31}$$

For a continuous map $f : T \to S$, define

$$f^* : \mathfrak{Ham}_F^M(S) \longrightarrow \mathfrak{Ham}_F^M(T), \qquad (f^*\Phi)_t = \Phi_{f(t)}. \tag{32}$$

Proposition 6.1 (Functoriality)**.** *The assignment $S \mapsto \mathfrak{Ham}_F^M(S)$ is a contravariant functor on profinite sets. The thermodynamic dynamics are natural under pullback:*

$$\tau_t^{(f^*\Phi)_u}(A) = \tau_t^{\Phi_{f(u)}}(A). \tag{33}$$

Proof. The norm bound is preserved by precomposition. Local continuity is preserved because a composite of continuous maps is continuous. The dynamics identity holds in every finite volume and therefore in the thermodynamic limit. Identity maps and composites act as required. □

Theorem 6.2 (Finite-cover descent)**.** *On a fixed small site of profinite sets with finite jointly surjective covers, $\mathfrak{Ham}_F^M$ is a sheaf of sets.*

Proof. Let $(S_i \to S)_{i=1}^k$ be a finite jointly surjective family and suppose $\Phi_i \in \mathfrak{Ham}_F^M(S_i)$ agree on every fiber product $S_i \times_S S_j$. For $s \in S$, choose a lift $s_i \in S_i$ and define $\Phi_s = (\Phi_i)_{s_i}$. Compatibility makes this independent of the lift.

The disjoint union $\coprod_i S_i \to S$ is a continuous surjection from a compact space to a Hausdorff space, hence a quotient map. Each finite-volume coefficient map descends to a continuous map on S. The common bound $\|\Phi_s\|_F \le M$ is inherited fiberwise. Thus the glued family belongs to $\mathfrak{Ham}_F^M(S)$. Uniqueness follows from joint surjectivity. □

Corollary 6.3 (Variable bounds)**.** *Let*

$$\mathfrak{Ham}_F^{\mathrm{ub}}(S) = \bigcup_{M>0} \mathfrak{Ham}_F^{M}(S). \tag{34}$$

Then $\mathfrak{Ham}_F^{\mathrm{ub}}$ also satisfies finite-cover descent.

Proof. A finite cover supplies finitely many bounds M_i. Their maximum is a global bound for the glued family. □

Remark 6.4 (What was proved). The sheaf statement is Proposed as an organization of a familiar analytic class. Its proof is elementary topology. The propagation theorem used to define the controlled class remains the established analytic input from Lieb-Robinson theory.

6.3 From a sheaf of sets to a moduli stack

Interactions may have symmetries, gauge identifications, and higher equivalences. Replacing equation (31) by a groupoid-valued or anima-valued object is natural. One might form an action groupoid for a chosen group of quasi-local automorphisms, then impose descent after specifying the topology on morphisms.

That construction is not carried out here. In particular, the following issues remain Open:

- which automorphisms are morphisms before imposing a spectral gap;
- how stabilization by product-state ancillas is represented;
- whether a chosen localization preserves the analytic subfunctor;
- how higher homotopies interact with a uniform locality witness;
- which universe and hypercompletion conventions define the condensed anima.

Our result supplies the object space and its dynamics. It is not yet the full condensed moduli stack.

7 Examples

7.1 Uniformly bounded finite-range couplings

Let $\Gamma = \mathbb{Z}^d$. Fix an interaction range R and a finite list of shapes $Z_1, \ldots, Z_N$. Suppose

$$\Phi_s(x + Z_j) = J_j(s, x) K_{j,x}, \tag{35}$$

where $\|K_{j,x}\| \leq 1$ and

$$\sup_{s,x,j} |J_j(s, x)| \leq J_*. \tag{36}$$

Only finitely many translates containing a given pair x, y contribute, and none contribute when $d(x, y)$ exceeds the maximal diameter of a shape. Consequently,

$$\sup_s \|\Phi_s\|_{F_a} \leq C(d, R, N, F_a) J_*. \tag{37}$$

If each $J_j(s, x)$ depends continuously on a finite cylinder of $s \in D^{\mathbb{Z}^d}$, the family is locally F-continuous. This gives a direct class of profinite points of equation (31).

7.2 Covariant disorder

Let $\Omega = D^{\mathbb{Z}^d}$ with shift T_x. A covariant local rule has the form

$$\Phi_\omega(x + Z) = U_x \Phi_{T_{-x}\omega}(Z) U_x^*, \tag{38}$$

where U_x translates observables. A uniform bound on the finite set of local coupling values gives uniform locality. Product-topology continuity gives local continuity. The global map $\omega \mapsto \Phi_\omega$ usually fails to be continuous in $\mathcal{B}_F$, since the norm takes a supremum over all translations.

The family theorem is designed for precisely this situation. It asks for the uniform bound directly and uses local continuity only where continuity is needed.

7.3 Time-dependent interactions

For completeness, let $u \mapsto \Phi_s(u)$ be strongly continuous in physical time and locally bounded in F-norm. On a compact time interval I, assume

$$\sup_{s \in S} \int_I \|\Phi_s(u)\|_F \, du < \infty. \tag{39}$$

The same proof yields a two-parameter cocycle $\tau^s_{t,r}$. In the time-independent exponent, replace $M|t - r|$ by

$$\int_{\min(t,r)}^{\max(t,r)} \|\Phi_s(u)\|_F \, du, \tag{40}$$

Thus $2C_F M|t - r|$ in the exponential becomes $2C_F \int_{\min(t,r)}^{\max(t,r)} \|\Phi_s(u)\|_F \, du$.

This extension is Established in [7]. We use time-independent notation elsewhere to keep the parameter s distinct from physical time.

8 Counterexamples and sharp distinctions

8.1 Pointwise finite norms do not give a common norm

Counterexample 8.1. Let $S = \mathbb{N} \cup \{\infty\}$ be the one-point compactification and let Ψ be a nonzero finite-range interaction. Put

$$\Phi_n = n\Psi, \qquad \Phi_\infty = 0. \tag{41}$$

Every fiber has finite F-norm, but $\sup_{s \in S} \|\Phi_s\|_F = \infty$.

This family is not locally continuous at infinity, so it is intentionally simple. It shows that a pointwise finiteness statement has the wrong quantifiers for uniform propagation.

8.2 Coefficientwise continuity needs a uniform tail condition

One can arrange interaction mass at increasing distances so that every fixed coefficient stabilizes while a global norm diverges. For example, choose pair terms on $\mathbb{Z}$ supported at $\{0, n\}$ with amplitudes tuned so that each interaction has finite norm but the norms grow with n. On the one-point compactification, coefficients on each fixed finite support converge to zero. No common Lieb-Robinson velocity follows.

This is the reason our local continuity definition is not presented as a substitute for equation (14). It is a second condition with a different job.

8.3 The moving weak-defect counterexample

The next example is central because it satisfies uniform locality and local continuity while its gaps collapse.

Counterexample 8.2 (Moving weak defect)**.** Let $\Gamma = \mathbb{N}$ with one qubit at each site. Write

$$p_x = |1\rangle\langle 1|_x. \tag{42}$$

For $n \geq 1$, define the on-site interaction

$$\Phi_n(\{x\}) = \begin{cases} n^{-1} p_n, & x = n, \\ p_x, & x \neq n, \end{cases} \qquad \Phi_n(Z) = 0 \quad \text{if } |Z| \neq 1. \tag{43}$$

Define the limiting interaction by

$$\Phi_\infty(\{x\}) = p_x. \tag{44}$$

Let $S = \mathbb{N} \cup \{\infty\}$. Then:

1. the family is finite-range and uniformly F-local;
2. $\Phi_n \to \Phi_\infty$ locally in F-norm;
3. all fibers have the same unique product ground state;
4. the GNS gaps are $\gamma_n = 1/n$ and $\gamma_\infty = 1$;
5. therefore every fiber is gapped but $\inf_{s \in S} \gamma_s = 0$.

Proof. Only on-site terms occur, so the interaction range is zero. With the usual normalization $F(0) > 0$,

$$\|\Phi_n\|_F = F(0)^{-1} \sup_x \|\Phi_n(\{x\})\| = F(0)^{-1}. \tag{45}$$

Thus the family has a common norm bound. For a fixed finite $\Lambda \subset \mathbb{N}$, the defect site n eventually lies outside Λ, so $\Phi_n|_\Lambda = \Phi_\infty|_\Lambda$. This proves local F-convergence.

The vector with every qubit in $|0\rangle$ is the unique zero-energy product ground state. In the nth fiber, flipping the qubit at site n costs $1/n$, while every other single flip costs 1. All multi-flip energies are sums of positive on-site costs. Hence the GNS gap is $1/n$. In the limiting fiber every single flip costs 1, so its gap is 1. More explicitly, the GNS Hamiltonian is diagonal on the dense basis of finite-spin-flip vectors. Its positive eigenvalues are finite sums of the on-site costs. Their infimum is therefore the least on-site cost, so no infrared sequence produces an additional spectral value below the stated gap. □

Remark 8.3. The limiting interaction in equation (44) does not omit the defect term. Instead, the weak term moves away and the local limit restores unit strength at every fixed site. This choice ensures that the limiting fiber is also gapped. It makes the counterexample stronger than one whose limiting fiber is simply gapless.

Corollary 8.4. *Uniform Lieb-Robinson estimates, local parameter continuity, and pointwise gappedness do not imply a parameter-uniform gap.*

Proof. The first two properties and failure of the third follow from theorem 8.2. □

8.4 Compact norm-continuous matrices and changing degeneracy

There is also a finite-dimensional obstruction. Let

$$S = \{0\} \cup \{1/n : n \geq 2\}, \qquad H_s = \mathrm{diag}(0, s, 1). \tag{46}$$

This is a norm-continuous family on a profinite compact set. At $s = 0$, the literal ground space has dimension two and the gap above it is 1. At $s = 1/n$, the ground state is unique and the literal ground-state gap is $1/n$. Every fiber is gapped, but there is no common lower bound.

If the two lowest levels are instead treated as one isolated ground band, the gap from that band to the level at 1 is at least $1/2$. Thus a later definition of a uniformly gapped substack must specify whether it tracks the literal ground eigenspace or an isolated low-energy band of constant rank.

9 Finite approximation and executable diagnostics

The companion Haskell program performs finite calculations. It is useful for checking constants and quantifiers, but it is not a proof assistant and does not establish a thermodynamic theorem.

9.1 Finite F-function diagnostics

On a finite line $\{0, \ldots, L-1\}$, the program computes

$$\|F\|_L = \max_x \sum_y F(|x-y|) \tag{47}$$

and

$$C_{F,L} = \max_{x,y} \frac{\sum_z F(|x-z|)F(|z-y|)}{F(|x-y|)}. \tag{48}$$

For increasing L, these values provide finite diagnostics for candidate F-functions. Bounded values in a numerical sample do not prove boundedness for the infinite lattice. An analytic estimate remains necessary.

9.2 Lieb-Robinson envelope

Given finite estimates $C_{F,L}$, M, supports X, Y, and time t, the code evaluates

$$\mathcal{E}_L(t; X, Y) = \frac{2}{C_{F,L}} \left(e^{2C_{F,L}M|t|} - 1\right) \sum_{x \in X, y \in Y} F(|x-y|). \tag{49}$$

Norms of A and B are set separately. This calculation is useful for catching missing factors and for seeing how a chosen decay profile affects the envelope.

9.3 Pointwise versus uniform predicates

For a finite list of positive gaps, both "every gap is positive" and "there is a positive minimum" hold. The distinction appears in a family of growing test sets. For the defect sequence,

$$\min_{1 \le n \le N} \gamma_n = \frac{1}{N}. \tag{50}$$

The program prints this value for increasing N. The analytic formula, not the finite run, proves that the infimum over all n is zero.

10 Interfaces required by later papers

10.1 Input to the gap paper

The thermodynamic gap is defined using the generator of the infinite-volume dynamics in a chosen ground-state representation. Paper 3 therefore needs the following outputs from the present paper:

1. the quasi-local algebra $\mathcal{A}_\Gamma$;
2. the class of admissible interactions $\mathcal{B}_F$;
3. a common locality witness $M_F(\Phi)$ for a family;
4. the thermodynamic automorphism group τ_t^s;
5. point-norm parameter continuity on local observables;
6. the distinction between local and global interaction topologies.

Paper 3 must add a ground-state convention and a gap witness. No gap predicate is part of theorem 4.2 or theorem 5.1.

10.2 Input to spectral flow

Quasi-adiabatic continuation assumes a differentiable interaction path, a uniform isolated spectral band, and stronger decay control on the interaction and its derivative. Under those hypotheses, Bachmann, Michalakis, Nachtergaele, and Sims construct a quasi-local spectral-flow automorphism [6].

The present paper supplies the propagation and thermodynamic-limit pattern used by that construction. It does not verify the spectral gap along a path. Accordingly, the implication

$$\text{uniformly local path} \Longrightarrow \text{spectral-flow equivalence} \tag{51}$$

is Obstructed without an independent uniform-gap assumption.

10.3 Input to observable algebras and K-theory

Paper 2 begins with the same quasi-local algebra and may form $C(S, \mathcal{A}_\Gamma)$ for a compact parameter space S. The existence of this C*-algebra of continuous functions is independent of the existence of a Hamiltonian dynamics. Conversely, the F-norm contains interaction data not recoverable from the abstract C*-algebra alone.

Thus the arrows

$$\Phi_s \longmapsto \mathcal{A}_\Gamma \longmapsto C(S, \mathcal{A}_\Gamma) \tag{52}$$

forget information. Any later K-theory invariant should be described as an invariant of the chosen observable-algebra construction, not as a complete encoding of the interaction.

11 Status ledger and limitations

Statement	Status	Reason
Finite-volume Lieb-Robinson estimate under finite F-norm	Established	Theorem 3.1 of [7], with earlier foundations in [4].
Uniform estimate for an S-family with a common F-norm bound	Established	Direct substitution of the common bound into the cited estimate.
Uniform thermodynamic dynamics	Established	Uniform version of the Cauchy-tail proof in [7].

Statement	Status	Reason
Parameter continuity from local F-continuity and a common tail bound	Established	Finite truncation plus uniform tails, compatible with the local F-convergence theorem of [7].
The functor $\mathfrak{Ham}_F^{\mathrm{ub}}$ on profinite sets	Proposed	Definition introduced here to organize the established analytic class.
Finite-cover descent for $\mathfrak{Ham}_F^{\mathrm{ub}}$	Established	Elementary gluing over quotient maps, once the analytic class is fixed.
Canonical condensed anima of all Hamiltonians and equivalences	Open	Morphisms, higher coherences, topology, stabilization, and size conventions remain to be specified.
Uniform locality from profiniteness alone	Obstructed	Profinite topology does not bound interaction norms.
Uniform gap from uniform locality and pointwise gaps	Obstructed	The moving weak-defect family in theorem 8.2.

11.1 Analytic limitations

We work with bounded interaction terms and finite-dimensional on-site spaces. Bosonic lattice models, continuum systems, and interactions with unbounded terms require different domains, representations, or state-dependent bounds. The conclusions should not be transferred to those settings by analogy.

The F-norm is sufficient, not necessary. Some models admit sharper or state-dependent propagation estimates even when the norm used here is infinite. Our functor therefore describes a robust controlled subspace of interactions, not every physically local model.

The light-cone estimate is an upper bound. It does not identify the exact front velocity, diffusion law, or transport coefficient. Disorder can produce anomalous or zero-velocity bounds that are stronger than the common estimate used here.

11.2 Categorical limitations

The sheaf in theorem 6.2 is set-valued. It does not encode unitary conjugacies, quasi-local automorphisms, defects, or higher families. Calling it a moduli stack without adding those morphisms would overstate the result.

The finite-cover sheaf property does not say that every invariant factors through a finite quotient of a profinite parameter. Locally constant families may approximate continuous families in a chosen norm, but an invariant must also be continuous in that norm before an approximation argument applies.

The associated condensed object depends on the selected topology on the interaction class. The global F-norm topology is convenient for analysis but too strong for many disorder families. The local F-topology admits those families but does not make the norm function locally bounded without a separate uniformity condition. This tradeoff is structural, not cosmetic.

11.3 Physical limitations

A Lieb-Robinson bound controls propagation of commutators. It does not imply a ground state, uniqueness, positivity of a gap, topological order, or invertibility. It also does not distinguish a trivial phase from a topological one.

The moving defect example shows that even an exceptionally simple family of commuting on-site Hamiltonians can violate uniform gappedness. Any proposed gapped substack must therefore carry the uniform gap as data or as a proved property with the correct quantifiers.

12 Discussion

The condensed perspective is most useful here as bookkeeping with teeth. It forces one to ask what an S-family is, which topology controls variation, and whether a statement is stable under pullback and finite-cover descent. Those questions expose a hidden ambiguity in the slogan "continuous family of local Hamiltonians." Continuity and locality are not single predicates.

The global interaction Banach topology makes compactness powerful. A continuous map from a compact S has a bounded image, so uniform propagation follows immediately. The same topology is poorly matched to covariant disorder because it monitors every translate at once. The local topology is better matched to physical observables, but compactness no longer supplies a global norm bound. The correct definition records both local continuity and uniform locality.

This two-part definition also behaves well under a finite profinite cover. Local continuity glues because continuous maps glue across a compact quotient. The uniform bound glues because a finite maximum exists. That is the precise sense in which condensed organization helps. It preserves a carefully defined analytic class under the covers used by the site.

The next step of the larger research program is harder. To define a uniformly gapped subfunctor, one must choose between finite-volume and GNS gaps, state the boundary conditions, track a ground band, and demand one gap constant for the whole parameter object. The counterexamples here show that none of this data can be suppressed.

There is also a useful lesson for formalization. A proof assistant interface should not encode "local interaction" as an unstructured label. It should carry an F-function, proofs of summability and convolution, an interaction, and a numerical norm witness. A family should carry a common witness rather than a function returning a separate witness for every point. That design makes the quantifier distinction visible in the type.

13 Conclusion

We have established a controlled first arrow

$$\text{local quantum interactions} \longrightarrow \text{parameterized thermodynamic dynamics.} \tag{53}$$

The analytic theorem is simple to state: a common interaction F-norm bound gives common Lieb-Robinson constants and uniform thermodynamic convergence. Local F-continuity then gives continuity of the dynamics on local observables. These results are inherited from established Lieb-Robinson theory with the family quantifiers made explicit.

We also proposed a profinite functor of controlled interaction families and proved finite-cover descent. This provides an honest object-level input for a future condensed moduli stack. Its categorical form does not produce the analytic estimates. It records the class for which those estimates have already been proved.

Finally, the moving weak defect shows why the next arrow cannot be automatic. Uniform locality and pointwise gaps coexist with a collapsing family gap. The uniformly gapped substack must therefore impose new analytic data. This boundary between organization and analysis is the main conclusion of the paper.

A Detailed comparison estimate

This appendix records the finite-volume comparison used in theorem 5.1. Let $\Lambda_m \subseteq \Lambda_n$ and write

$$H_{\Lambda_n} = H_{\Lambda_m} + V_{m,n}, \tag{54}$$

where $V_{m,n}$ contains terms not wholly supported in Λ_m. Duhamel's formula gives

$$\begin{aligned} &\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \\ &\quad = i \int_0^t \tau_r^{\Lambda_n}\left([V_{m,n}, \tau_{t-r}^{\Lambda_m}(A)]\right) dr. \end{aligned} \tag{55}$$

Taking norms and decomposing $V_{m,n}$ into interaction terms yields

$$\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \leq \sum_{Z \in \mathcal{S}_{m,n}} \int_0^{|t|} \|[\Phi(Z), \tau_{|t|-r}^{\Lambda_m}(A)]\| dr, \tag{56}$$

where $\mathcal{S}_{m,n}$ is the set of new terms meeting the complement of Λ_m. The Lieb-Robinson estimate bounds each commutator by a spatial sum from X to Z. Summing over Z and using the interaction norm gives a tail from X to $\Lambda_n \setminus \Lambda_m$. Integrating the time exponential gives equation (26), up to the harmless choice between $e^{2C_F M|t|} - 1$ and its upper bound $2C_F M|t|e^{2C_F M|t|}$.

Every constant in this argument is controlled by F, C_F, M, $|X|$, and the time interval. This is the entire reason the thermodynamic limit is uniform in the family parameter.

B A finite-cover gluing lemma

Lemma B.1. *Let $q : K \to S$ be a continuous surjection from a compact space to a Hausdorff space. Let Y be any topological space. A map $f : S \to Y$ is continuous if and only if $f \circ q$ is continuous.*

Proof. The map q is closed because images of compact sets are compact and compact subsets of a Hausdorff space are closed. A continuous closed surjection is a quotient map. The assertion is the defining property of the quotient topology. $\square$

Remark B.2. In theorem 6.2, take $K = \coprod_i S_i$. The target Y is the finite-volume interaction space for each fixed Λ. The lemma proves continuity of the glued local restrictions. The analytic norm bound is checked separately.

C Logical form of uniform conditions

The following formulas summarize the quantifiers that the paper keeps separate.

Pointwise locality is

$$\forall s \in S \; \exists M_s < \infty : \|\Phi_s\|_F \leq M_s. \tag{57}$$

Uniform locality is

$$\exists M < \infty \; \forall s \in S : \|\Phi_s\|_F \leq M. \tag{58}$$

Pointwise gappedness is

$$\forall s \in S \; \exists \gamma_s > 0 : \operatorname{gap}(H_s) \geq \gamma_s. \tag{59}$$

Uniform gappedness is

$$\exists \gamma > 0 \; \forall s \in S : \operatorname{gap}(H_s) \geq \gamma. \tag{60}$$

Neither exchange of quantifiers is a matter of notation. Compactness permits the exchange only when the relevant bound or gap function has suitable continuity properties. The ground-state gap can fail to be lower semicontinuous when ground-state multiplicity changes. The moving defect shows failure even with a fixed unique product ground state if the interaction topology is local.

D Notation and reproducibility

Symbol	Meaning
Γ	Countable metric lattice or graph.
$\mathcal{A}_\Lambda$	Matrix algebra of observables in finite volume Λ.
$\mathcal{A}_\Gamma$	Norm-completed quasi-local C*-algebra.
Φ	Bounded self-adjoint interaction.
F	Uniformly summable decay function with a convolution bound.
C_F	Convolution constant of F.
$\|\Phi\|_F$	Interaction norm in equation (11).

Symbol	Meaning
S	Compact parameter space, often profinite.
$M_F(\Phi)$	Uniform interaction-norm bound over S.
τ_t^{Λ,Φ_s}	Finite-volume dynamics in fiber s.
τ_t^s	Thermodynamic dynamics in fiber s.
$\mathfrak{Ham}_F^M$	Proposed sheaf of locally continuous families with common bound M.

The Haskell sources accompanying this article are in the repository directory

`src/locality-condensed-families/`.

They compile with GHC using `-Wall`, `-Wextra`, and `-Werror`. All exported functions have explicit type signatures. The computations are deterministic and require no external packages beyond `base`.

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