Physical Realizability of Invertible Phase Classes: Microscopic Witnesses, Bordism Targets, and Relative Charges

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14 July 2026

Abstract

An abstract class in bordism, stable homotopy, or generalized cohomology is not yet a quantum material. A microscopic realization also needs a local Hilbert space, an interaction, a symmetry action, a ground-state convention, and a spectral gap bounded below independently of system size. This paper turns that observation into a realizability matrix. Each row records an explicit Hamiltonian, its symmetry, the theorem or exact calculation behind its uniform gap, a microscopic invariant, and the status of its comparison with an abstract class.

Three one-dimensional tests are worked out. The class-D Kitaev chain has an exactly flat bulk spectrum at its solvable point and realizes the nontrivial fermionic $\mathbb{Z}_2$ phase. A stack of BDI chains realizes the free integer index, while the Fidkowski-Kitaev interaction identifies indices differing by eight. The spin-1 AKLT chain has a rigorous bulk gap and carries the nontrivial projective $SO(3)$ edge representation. These examples verify specific low-dimensional rows. They do not prove that every class counted by a field-theory formula has a lattice representative.

We keep three spectra separate. Freed and Hopkins classify deformation classes of reflection-positive invertible field theories under stated field-theoretic hypotheses. Kubota constructs an operator-algebraic Ω-spectrum from invertible gapped quantum spin systems. A condensed phase spectrum remains a proposed organization of parameterized microscopic families. No comparison theorem presently identifies these objects.

Finally, for a family on B with gapless locus Σ and $U = B \setminus \Sigma$, we state the exact hypotheses under which the connecting morphism $E^q(U) \to E^{q+1}(B, U)$ defines a relative class, localizes by excision, and becomes a Thom class on Σ. Its interpretation as a physical transition charge still requires a specified microscopic comparison map. The accompanying Haskell registry checks witness completeness, dimension bookkeeping, and the degrees in this relative-charge construction.

Contents

1 The realization problem

Classifications of invertible phases appear in several mathematical forms. Free fermions lead to operator K-groups. Low-energy invertible field theories lead to bordism and Anderson-dual groups. Microscopic spin systems can themselves be organized into homotopy types. These outputs look similar because stacking supplies addition and spatial suspension shifts degrees. They are nevertheless built from different objects and different notions of homotopy.

The question addressed here is deliberately concrete:

> Given an abstract invertible class, is there a local symmetric Hamiltonian whose thermodynamic phase maps to that class, with a gap controlled uniformly in volume?

The phrase "maps to" is essential. A table in which two groups happen to be isomorphic does not supply a map between them. Nor does it prove that an abstract generator has a representative satisfying a chosen microscopic notion of locality.

1.1 Spatial dimension and spacetime dimension

Throughout the paper,

$$d = \text{spatial dimension}, \qquad n = d + 1 = \text{spacetime dimension}. \tag{1}$$

Hamiltonians and lattices are indexed by d. The tangential symmetry type in the Freed-Hopkins formula is indexed by n. Thus every comparison row must visibly apply the shift in equation (1). In particular, a one-dimensional chain has $d = 1$ but contributes to a two-dimensional spacetime theory with $n = 2$.

1.2 What this paper proves

The main result is a verified portfolio, not a surjectivity theorem.

Theorem 1.1 (Low-dimensional witness portfolio)**.** *With the boundary conventions stated below, the following systems satisfy the microscopic columns of the realizability matrix:*

1. *the solvable class-D Kitaev chain realizes the nontrivial free and interacting one-dimensional fermionic $\mathbb{Z}_2$ class;*
2. *r copies of the BDI chain realize free index r, and the symmetry-preserving interacting classification depends on r modulo eight;*
3. *the spin-1 AKLT chain realizes the nontrivial $SO(3)$-protected bosonic one-dimensional phase detected by its spin-$1/2$ boundary representation.*

For each row there is an explicit finite-range interaction, an explicit symmetry, a uniform bulk-gap calculation or theorem, and a microscopic invariant. The corresponding bordism or field-theory comparison is a low-dimensional agreement using an effective-theory interpretation. The theorem does not construct a general map from microscopic phases to the Freed-Hopkins spectrum.

The proof occupies sections 4 to 6. The last sentence is part of the theorem's scope, not a disclaimer added after the fact.

1.3 Status language

We use four statuses for realization questions.

Definition 1.2 (Realization status)**.** Fix the microscopic rules, including locality, symmetry, stabilization, boundary convention, and target invariant.

- Realized means an explicit interaction, symmetry action, uniform-gap verification, and invariant computation are all present.
- Candidate means a model is explicit but at least one required witness is missing.
- Open means no realization or obstruction theorem is asserted.
- Obstructed means a theorem rules out realization under the fixed rules.

The absence of a known model gives status Open, never Obstructed.

2 Three phase objects that must remain distinct

2.1 Microscopic stabilized phases

Let Γ be a lattice of bounded geometry. Each site x carries a finite-dimensional Hilbert space $\mathcal{H}_x$, or a finite-dimensional graded local algebra in a fermionic formulation. A finite-range or uniformly almost-local interaction Φ assigns a self-adjoint local operator $\Phi(X)$ to each finite $X \Subset \Gamma$.

Definition 2.1 (Microscopic phase datum)**.** A microscopic datum in spatial dimension d is a tuple

$$W = (\Gamma, \{\mathcal{H}_x\}, \Phi, G, \alpha, \omega, \gamma, \nu), \tag{2}$$

where G acts on the quasi-local algebra by α, Φ is G-invariant, ω specifies the ground-state sector, $\gamma > 0$ is a volume-independent bulk-gap lower bound, and ν is a phase invariant with a declared codomain. Fermionic parity is included in the symmetry data when appropriate.

For a periodic sequence of finite volumes Λ_L, the uniform-gap condition used in the examples is

$$\inf_{L \geq L_0} \left(E_{m_L+1}(H_{\Lambda_L}) - E_{m_L}(H_{\Lambda_L})\right) \geq \gamma > 0. \tag{3}$$

Here m_L is the dimension of the prescribed low-energy sector. For a periodic chain with a unique ground state, $m_L = 1$. For an open AKLT chain, the four edge states form the low-energy sector and the relevant bulk gap is the gap above that sector.

Two data represent the same stabilized phase if, after adding atomic product-state ancillas, they are joined by a symmetry-preserving path of interactions that remains uniformly local and satisfies equation (3) along the path. Invertibility means that stacking with some second datum is gapped-homotopic, after stabilization, to the chosen atomic phase.

Remark 2.2. An additive numerical invariant does not prove invertibility. Invertibility requires an inverse under stacking in the microscopic phase space.

2.2 Freed-Hopkins deformation classes

Fix a spacetime symmetry type (H_n, ρ_n), where $\rho_n : H_n \to O_n$ has compact internal kernel. Let MTH be its Madsen-Tillmann spectrum and let $I\mathbb{Z}(1)$ denote the indicated Anderson-dual target.

Theorem 2.3 (Freed-Hopkins, discrete topological sector)**.** Established *Deformation classes of reflection-positive invertible extended topological field theories of the fixed symmetry type are computed, in the discrete topological sector, by the torsion subgroup*

$$\mathfrak{F}\mathfrak{H}_n(H) := [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tors}}. \tag{4}$$

The application of this result to lattice phases assumes the existence and validity of an appropriate low-energy effective field theory.

The theorem is a field-theory classification. It is not a theorem that each element of equation (4) has a finite-range lattice Hamiltonian. The broader extension to all nontopological reflection-positive invertible theories is also not silently imported here.

2.3 Kubota's microscopic Omega-spectrum

Theorem 2.4 (Kubota)**.** ESTABLISHED *There is an Ω-spectrum IP_* constructed from invertible gapped quantum spin systems on Euclidean spaces. Its homotopy groups recover the corresponding smooth homotopy groups of microscopic invertible systems. The construction uses operator algebras, uniformly almost-local interactions, relaxed lattices, stabilization by atomic systems, and sheaves of smooth parameter families. It also has variants for crystallographic spatial symmetry.*

The objects in this theorem include a chosen nondegenerate gapped GNS ground state. Compact Lie on-site symmetry yields the stated naive equivariant variant. This should not be upgraded without proof to a genuine representation-graded equivariant spectrum.

2.4 The proposed condensed spectrum

A condensed moduli stack would evaluate parameterized Hamiltonian families on profinite or more general condensed test objects. Its invertible sector could then be group-completed and stabilized. This is a useful proposed target for disorder and inverse-limit parameter spaces. It is not Kubota's smooth-manifold sheaf and it is not the bordism spectrum in theorem 2.3.

The current comparison situation is therefore

$$\begin{tikzcd} \mathfrak{Inv}(\mathfrak{Mic}^G_d) \arrow[r, dashed, "\text{partial microscopic packaging}"] \arrow[dr, dotted, "\text{low-energy comparison}"'] & IP^G_d \arrow[d, dotted, "\text{comparison open}"] \\ & \mathfrak{FH}_{d+1}(H) \end{tikzcd} \qquad IP^G_d \overset{\text{no theorem}}{\dashrightarrow} \mathfrak{Cond}^G_d. \tag{5}$$

Solid arrows would require definitions on a common domain and proofs of continuity, symmetry compatibility, stabilization compatibility, and homotopy invariance. The dotted arrows in equation (5) are research questions.

3 The realizability matrix

3.1 Witness columns

Definition 3.1 (Realizability row)**.** A realizability row for a proposed class c contains the following fields:

1. spatial dimension d and spacetime dimension $n = d + 1$;
2. microscopic kinematics and boundary convention;
3. a local interaction Φ;

4. an exact symmetry action and its square or extension law;
5. a uniform-gap witness, either an exact calculation or a cited theorem;
6. a microscopic invariant ν_{mic};
7. an abstract target and a stated comparison status;
8. one of the four realization statuses.

The row is complete only if every required field is populated. Numerical evidence from finitely many lengths is not a uniform-gap theorem.

3.2 The verified low-dimensional matrix

System	Dimensions	Gap witness	Microscopic invariant	Abstract comparison
Class-D Kitaev chain	$d = 1$, $n = 2$	Exact flat BdG spectrum at the solvable periodic point	Pfaffian parity -1, equivalently one Majorana per open end	Spin/Arf $\mathbb{Z}_2$ agreement in the effective-theory sector; no general spectrum map
BDI stack	$d = 1$, $n = 2$	Exact free bulk gap plus the Fidkowski-Kitaev symmetric interacting path for eight copies	Free index r, interacting class r mod 8	Pin^- and Arf-Brown-Kervaire $\mathbb{Z}_8$ agreement; comparison uses the field-theory interpretation
Spin-1 AKLT chain	$d = 1$, $n = 2$	Rigorous AKLT bulk-gap theorem	Nontrivial projective $SO(3)$ edge representation	Agreement with the $H^2(SO(3), U(1)) \cong \mathbb{Z}_2$ sector; no general bordism realization theorem

Each row has microscopic status Realized. Its last column states a more limited comparison claim. The word "agreement" does not mean that the three spectra in equation (5) have been identified.

Proposition 3.2 (Matrix verification criterion)**.** *A row has status* Realized *if the interaction and symmetry are explicit, the bulk gap satisfies equation (3), the microscopic invariant is computed and stable under the chosen phase relation, and the row claims no abstract comparison stronger than the supplied map or effective-theory argument.*

Proof. This is a contract rather than a classification theorem. The first three conditions produce a microscopic witness of the claimed phase. The last condition prevents a microscopic witness from being re-labelled as a surjectivity theorem for a different classification object. □

4 The class-D Kitaev chain

4.1 Hamiltonian and symmetry

On a chain of L complex fermions, let c_j, c_j^* satisfy the canonical anticommutation relations. With periodic or open boundary convention as specified, set

$$\begin{aligned} H_D(\mu, t, \Delta) = & -\mu \sum_j \left(c_j^* c_j - \frac{1}{2} \right) - t \sum_j \left(c_j^* c_{j+1} + c_{j+1}^* c_j \right) \\ & + \Delta \sum_j \left(c_j c_{j+1} + c_{j+1}^* c_j^* \right), \end{aligned} \tag{6}$$

where $t, \Delta \in \mathbb{R}$. Fermion parity $(-1)^F$ is exact. In BdG form, particle-hole conjugation is intrinsic and squares to $+1$. No time-reversal symmetry is imposed, so the system is in class D.

Introduce Majorana operators

$$a_j = c_j + c_j^*, \qquad b_j = \frac{c_j - c_j^*}{i}. \tag{7}$$

At the solvable point $\mu = 0$ and $\Delta = t > 0$, an open chain has

$$H_D = it \sum_{j=1}^{L-1} b_j a_{j+1}, \tag{8}$$

up to the conventional overall normalization. The operators a_1 and b_L do not occur in equation (8). They form the two boundary Majorana zero modes.

4.2 Uniform periodic gap

For a periodic chain, the BdG symbol may be chosen as

$$h_D(k) = (-\mu - 2t \cos k)\tau_z + 2\Delta \sin k \, \tau_y. \tag{9}$$

Here the Nambu spinor is $\Psi_k = (c_k, c_{-k}^*)^\mathsf{T}$. The matrices τ_y and τ_z are Pauli matrices in this ordered particle-hole basis, with the pairing phase chosen so that real Δ multiplies τ_y.

Its positive quasiparticle energy is

$$E(k) = \sqrt{(\mu + 2t \cos k)^2 + 4\Delta^2 \sin^2 k}. \tag{10}$$

Proposition 4.1 (Exact solvable-point gap)**.** *At $\mu = 0$ and $\Delta = t > 0$, every allowed periodic momentum satisfies $E(k) = 2t$. Hence the periodic many-body bulk gap is bounded below by $2t$ for every L.*

Proof. Substitution in equation (10) gives

$$E(k)^2 = 4t^2 \cos^2 k + 4t^2 \sin^2 k = 4t^2. \tag{11}$$

A quadratic BdG Hamiltonian diagonalizes into independent positive-energy quasiparticles. The least excitation energy is therefore $2t$, independent of the momentum mesh and of L. If one restricts the periodic Hilbert space to a fixed even-fermion-parity sector, a single quasiparticle is excluded and the first allowed excitation at this point has two quasiparticles and energy $4t$. The stated $2t$ volume-independent lower bound is valid with or without that restriction. □

4.3 The microscopic invariant

At the particle-hole fixed momenta $k = 0, \pi$, the class-D invariant can be written as the sign of a product of Pfaffians. In the convention of equation (9),

$$\nu_D = \operatorname{sgn}\big((\mu + 2t)(\mu - 2t)\big) \in \{+1, -1\}. \tag{12}$$

Thus $|\mu| < 2|t|$ gives $\nu_D = -1$. The solvable point lies in this region. The open-chain boundary Majoranas supply a second diagnostic of the same phase.

Corollary 4.2 (Class-D row)**.** *The tuple consisting of equation (6), fermion parity and BdG particle-hole structure, periodic boundary conditions for the bulk gap, open boundary conditions for the edge diagnostic, the lower bound $2t$, and $\nu_D = -1$ is a* Realized *microscopic row.*

4.4 Comparison status

In spacetime dimension $n = 2$, the corresponding invertible spin field theory is the nontrivial Arf theory. Both the microscopic interacting class-D classification and this low-dimensional field-theory sector are $\mathbb{Z}_2$. The Kitaev chain is the standard microscopic representative of the nontrivial element.

This agreement is strong evidence for the expected low-energy comparison. It does not define the dotted arrow in equation (5) on all microscopic phases, parameter families, or dimensions.

5 BDI and the interaction reduction

5.1 Free integer index

Take r copies of the real Kitaev chain. Let the antiunitary symmetry T act by complex conjugation in the real-fermion basis, so that

$$TiT^{-1} = -i, \qquad Tc_{j,\alpha}T^{-1} = c_{j,\alpha}, \qquad T^2 = +1. \tag{13}$$

Together with BdG particle-hole structure this is class BDI. The real off-diagonal symbol has an integer winding number. A stack of r solvable chains has

$$\nu_{\text{free}} = r \in \mathbb{Z} \tag{14}$$

and carries r Majorana zero modes at each open end. Its periodic bulk gap is again $2t$.

5.2 Why the integer is not an interacting invariant

Theorem 5.1 (Fidkowski-Kitaev reduction)**.** Established *For one-dimensional BDI fermions with the symmetry in equation (13), symmetry-preserving local interactions reduce the free $\mathbb{Z}$ classification to $\mathbb{Z}_8$. Eight boundary Majorana modes can be gapped without a fermion bilinear and without breaking the symmetry. Correspondingly, eight copies of the nontrivial free chain are connected to the trivial interacting phase by a symmetry-preserving gapped path.*

The interaction is local and quartic in the boundary Majoranas, and its bulk extension supplies the gapped interpolation. The cited construction, not a finite-size numerical extrapolation, is the gap witness for the reduction. For eight stacked chains, write γ_a for the eight unpaired boundary operators selected from the a_j, b_j basis used in equation (8). At an end where these γ_a are T-even, a monomial $\gamma_a\gamma_b\gamma_c\gamma_d$ with a real coefficient is also T-even and preserves fermion parity. The Fidkowski-Kitaev interaction is built from such quartic terms, so the mechanism gaps the boundary multiplet without violating equation (13).

Proposition 5.2 (Interacting BDI row)**.** *The microscopic invariant of the interacting BDI stack is*

$$\nu_{\text{int}} = r \pmod 8. \tag{15}$$

Rows r and $r+8$ represent the same stabilized interacting phase, whereas their free integer invariants differ.

Proof. The equivalence $r \sim r+8$ follows by stacking the Fidkowski-Kitaev gapped trivialization of eight chains. Their interacting classification also distinguishes the remaining residue classes, for example through the symmetry action and algebra of boundary degrees of freedom. Hence the quotient is precisely $\mathbb{Z}_8$. □

Counterexample 5.3 (Free K-theory is not the universal interacting answer)**.** The free phases with indices 0 and 8 are distinct in the integer classification. By theorem 5.1, they are equal after symmetry-preserving interactions are admitted. Therefore a free K-group cannot be identified with the interacting microscopic phase group in general.

5.3 Pin structure and the abstract class

The antiunitary relation $T^2 = +1$ for the BDI fermion corresponds to the Pin^- tangential structure in two spacetime dimensions. The Arf-Brown-Kervaire invariant has values in $\mathbb{Z}_8$, matching equation (15). This is a particularly sharp low-dimensional agreement between microscopic interaction reduction and bordism data.

The comparison still uses the low-energy field-theory interpretation in theorem 2.3. It is not a construction of a natural equivalence between Kubota's IP_* and the Freed-Hopkins mapping spectrum.

6 The SO(3)-protected AKLT chain

6.1 Interaction and symmetry

At each site place the spin-1 irreducible representation of $SO(3)$. Write $\mathbf{S}_j = (S_j^x, S_j^y, S_j^z)$. For neighboring sites let $P^{(2)}_{j,j+1}$ be the orthogonal projection onto total spin two. Since $x = \mathbf{S}_j \cdot \mathbf{S}_{j+1}$ has eigenvalues $-2, -1, 1$ in total-spin sectors zero, one, and two,

$$P^{(2)}_{j,j+1} = \frac{1}{6}\left((\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 + 3\mathbf{S}_j \cdot \mathbf{S}_{j+1} + 2\right). \tag{16}$$

The AKLT Hamiltonian is

$$H_{\mathrm{AKLT}} = \sum_j P^{(2)}_{j,j+1}. \tag{17}$$

Every term is a positive nearest-neighbor projection. The diagonal on-site $SO(3)$ action commutes with each term.

6.2 Ground states, edges, and bulk gap

The valence-bond construction represents each spin one as the symmetric subspace of two virtual spin-1/2 variables and pairs neighboring virtual spins into singlets. On a periodic chain it gives the unique ground state. On an open chain, one virtual spin-1/2 remains at each boundary, giving a four-dimensional ground-state sector.

Theorem 6.1 (AKLT bulk gap)**.** Established *The one-dimensional AKLT interaction has a strictly positive bulk spectral gap. Equivalently, the periodic finite-volume gaps have a positive lower bound for sufficiently large volume, and the open-chain spectrum has a positive gap above its edge-state ground sector.*

This is the imported gap theorem for the row. Frustration freeness alone would not have been enough.

6.3 Projective boundary invariant

The physical spin-1 representation descends from $SU(2)$ to a linear representation of $SO(3)$. A single virtual edge spin transforms in the spin-1/2 representation of $SU(2)$. The central element $-1 \in SU(2)$ acts on it as $-\mathrm{id}$, so this action does not descend to a linear representation of $SO(3)$. It defines the nontrivial projective class

$$[\omega_{\mathrm{edge}}] \in H^2(SO(3), U(1)) \cong \mathbb{Z}_2. \tag{18}$$

Proposition 6.2 (Stability of the AKLT edge class)**.** *Along a uniformly gapped $SO(3)$-symmetric path of one-dimensional spin interactions, the projective equivalence class of the boundary representation is constant.*

Proof sketch. Quasi-adiabatic spectral flow transports the ground-state sector by a quasi-local automorphism that intertwines the symmetry. The induced half-chain boundary representations are therefore unitarily equivalent up to tensoring with linear on-site representations. Such tensor factors do not change the class in $H^2(SO(3), U(1))$. The operator-algebraic version is proved for compact symmetry groups by Bachmann and Nachtergaele. □

Corollary 6.3 (AKLT row)**.** *The interaction equation (17), diagonal spin-1 $SO(3)$ action, AKLT bulk-gap theorem, periodic and open boundary conventions, and nontrivial class equation (18) form a* REALIZED *microscopic row.*

The same $\mathbb{Z}_2$ appears in the low-dimensional bosonic field-theory sector with $SO(3)$ background. As in the fermionic examples, this row checks a generator and not the essential surjectivity of a general comparison functor.

7 Relative transition charge

7.1 The exact sequence of a pair

Let E be a multiplicative generalized cohomology theory represented by a spectrum. Let B be a space in a category where the pair axiom and excision hold, and let $\Sigma \subset B$ be closed. Put $U = B \setminus \Sigma$. We assume the inclusions have been replaced by cofibrant pairs when necessary, so that the cofiber model computes relative cohomology.

The pair (B, U) has the long exact sequence

$$\cdots \longrightarrow E^q(B) \xrightarrow{j^*} E^q(U) \xrightarrow{\partial} E^{q+1}(B, U) \xrightarrow{i^*} E^{q+1}(B) \longrightarrow \cdots . \tag{19}$$

Definition 7.1 (Relative class)**.** For $\nu \in E^q(U)$, define

$$Q_\Sigma(\nu) := \partial\nu \in E^{q+1}(B, U). \tag{20}$$

As algebraic topology this is an ESTABLISHED connecting class. Calling it a physical transition charge is PROPOSED until ν is tied to a microscopic family by a specified comparison map.

Proposition 7.2 (Extension criterion)**.** *The class $Q_\Sigma(\nu)$ vanishes if ν is the restriction of a class in $E^q(B)$. Conversely, exactness implies that $Q_\Sigma(\nu) = 0$ precisely when ν lies in the image of j^*.*

Proof. Exactness at $E^q(U)$ gives $\ker \partial = \operatorname{im} j^*$. □

The converse concerns extension in the chosen cohomology theory. It does not say that a Hamiltonian family itself extends across Σ while remaining gapped.

7.2 Excision near the gapless locus

Theorem 7.3 (Localization by excision). *Assume B is paracompact Hausdorff and has the homotopy type required by E. Let N be an open neighborhood of Σ such that excision applies to the pair (B, U). Then inclusion of pairs induces an isomorphism*

$$E^{q+1}(B, B \setminus \Sigma) \xrightarrow{\cong} E^{q+1}(N, N \setminus \Sigma). \tag{21}$$

Proof. Choose N so that the closure of $B \setminus N$ is contained in the open set U. Excision removes $B \setminus N$ from both entries of the pair. The resulting pair is $(N, N \setminus \Sigma)$. Generalized cohomology represented on cofibrant pairs satisfies this excision axiom. □

This theorem is the precise sense in which equation (20) is supported near Σ.

7.3 Thom identification

Suppose now that B is a smooth manifold and $\Sigma \hookrightarrow B$ is a closed smooth submanifold of codimension c. Let $\mathcal{N} \to \Sigma$ be its normal bundle. Assume $\mathcal{N}$ is E-oriented, meaning it has a Thom class

$$u_E \in \widetilde{E}^c(\mathrm{Th}(\mathcal{N})) \tag{22}$$

whose restriction to every normal fiber is a generator in the required sense.

Theorem 7.4 (Relative class on the critical locus). *Under the excision and orientation hypotheses above, a tubular neighborhood and the Thom isomorphism give*

$$\begin{aligned} E^{q+1}(B, B \setminus \Sigma) &\cong E^{q+1}(N, N \setminus \Sigma) \\ &\cong \widetilde{E}^{q+1}(\mathrm{Th}(\mathcal{N})) \\ &\cong E^{q+1-c}(\Sigma). \end{aligned} \tag{23}$$

Thus $Q_\Sigma(\nu)$ determines a localized class of degree $q + 1 - c$ on Σ.

Proof. The first isomorphism is theorem 7.3. Radial deformation in a tubular neighborhood identifies the pair with the disk and sphere bundles $(D\mathcal{N}, S\mathcal{N})$, whose quotient is the Thom space. Cup product with u_E is the Thom isomorphism and shifts degree by c. □

Without an E-orientation, the last line must use the corresponding twisted cohomology. Omitting that twist can change or destroy the purported charge.

7.4 Linking spheres

At a point $x \in \Sigma$ where the normal bundle is locally trivial, restrict the Thom representative to a normal disk D^c and its boundary S^{c-1}. The resulting local class measures the obstruction on a linking sphere. For ordinary cohomology this is the familiar local degree. For K-theory or another spectrum it is the corresponding generalized cohomology class.

Remark 7.5. A linking sphere must live in the full parameter space relevant to the invariant. For a band Hamiltonian this may include momentum together with a tuning parameter. A loop in the tuning parameter alone can miss the charge. For example, an isolated codimension-three Weyl-type degeneracy is linked by an S^2. Its three normal coordinates may be three momentum components, or two momenta together with a tuning mass in a parameterized two-dimensional model.

7.5 Naturality required for physics

Suppose a microscopic family H on U has an invariant $\nu_{\mathrm{mic}}(H)$. To interpret equation (20) physically, one needs a natural transformation

$$\kappa : \nu_{\mathrm{mic}}(H) \longmapsto \nu_E(H) \in E^q(U) \tag{24}$$

that is invariant under the chosen gapped homotopies and compatible with restriction to open sets. Naturality then makes the connecting morphism commute with the microscopic restriction maps.

Proposition 7.6 (Conditional physical charge)**.** *If the map equation (24) exists with these properties, then $Q_\Sigma(\nu_E(H))$ is an invariant of the microscopic gapped family on U and is supported near the gapless locus by theorem 7.3.*

Proof. Homotopy invariance of κ makes $\nu_E(H)$ a phase invariant. Naturality of the long exact sequence transports this invariance through ∂. Excision supplies the support statement. □

The proposition is conditional. A relative group by itself does not create the microscopic map κ.

8 Failures that a realizability claim must avoid

Counterexample 8.1 (Equal groups without a comparison)**.** Suppose two independently defined phase theories both produce $\mathbb{Z}_8$. Choosing generators gives many abstract group isomorphisms. None of those choices proves that a microscopic Hamiltonian flows to a particular field theory, respects families, or commutes with stacking.

Counterexample 8.2 (A finite-size gap table)**.** Assume exact diagonalization gives gaps above 0.2 for lengths up to twenty. The sequence could still decrease to zero at larger lengths. The table is evidence, not a witness for equation (3).

Counterexample 8.3 (Edge modes without a bulk convention)**.** An open finite chain can have nearly zero-energy boundary states for reasons unrelated to a stable bulk phase. Without a periodic bulk-gap statement and a symmetry-stable boundary invariant, zero modes alone do not complete a realizability row.

Counterexample 8.4 (Pointwise representatives without a family)**.** Choosing one Hamiltonian for each abstract class does not automatically give a continuous map of phase spaces or spectra. Parameter continuity and uniform locality and gap bounds are additional data.

Remark 8.5 (Open is not obstructed). If no lattice model is known for a bordism class, the correct status is Open. Status Obstructed requires a theorem such as an anomaly, locality, or finite-dimensionality obstruction formulated for the same microscopic rules.

9 What condensed organization contributes

Condensed mathematics can organize families over profinite sets, disorder hulls, and inverse limits. It can impose descent on compatible local data and make stacking functorial. For the realizability problem, a condensed test object could record how a portfolio of witnesses varies over a totally disconnected compact parameter space.

It does not provide any missing witness in equation (2). In particular, sheaf descent does not prove the operator inequality behind a gap, construct an inverse phase, or identify a low-energy field theory.

Definition 9.1 (Proposed condensed realizability subfunctor)**.** For a condensed test object S, let $\mathcal{R}_d^G(S)$ consist, when defined, of S-families of microscopic witnesses with a common locality bound, a common positive gap, continuous local coefficients, and a locally constant realizability status. Pullback is by parameter restriction.

This definition is Proposed. A full construction must specify higher isomorphisms, stabilization, and the topology on symmetry actions. Even after construction, a morphism $\mathcal{R}_d^G \to \mathfrak{FH}_{d+1}(H)$ remains a separate comparison problem.

10 Executable witness registry

The Haskell program under

`src/physical-realizability/`

implements a finite registry rather than a symbolic classifier. A row has separate fields for d, n, interaction, symmetry, gap evidence, microscopic invariant, abstract comparison, and status. Validation checks that $n = d + 1$, that a Realized row has every microscopic witness, and that no row describes effective-theory agreement as a general spectrum equivalence.

The program also implements the degree bookkeeping

$$q \overset{\partial}{\mapsto} q + 1 \xmapsto{\text{Thom, codim } c} q + 1 - c. \tag{25}$$

It records whether extension forces the connecting class to vanish and whether an E-orientation is available for the Thom step.

These are contract checks. A Boolean value cannot prove the AKLT gap or the Fidkowski-Kitaev interpolation. The registry stores those results as cited theorem witnesses and rejects rows in which the citation field is absent.

11 Status ledger

Statement	Status	Basis
Freed-Hopkins classification of discrete reflection-positive invertible TFTs	ESTABLISHED	Stable homotopy and bordism theorem under its field-theory hypotheses
Kubota Ω-spectrum for invertible gapped spin systems	ESTABLISHED	Operator-algebraic construction with smooth families and almost-local interactions
Kitaev class-D row	REALIZED	Explicit interaction, exact periodic gap, Pfaffian invariant, edge Majoranas
BDI mod-eight row	REALIZED	Explicit free stacks and Fidkowski-Kitaev interacting reduction
AKLT $SO(3)$ row	REALIZED	Explicit projector interaction, rigorous bulk gap, projective edge class
General microscopic to Freed-Hopkins comparison	OPEN	No map with all required functorial and analytic properties is supplied
Identification of Kubota and Freed-Hopkins spectra	OPEN	Domains and homotopies differ; no equivalence theorem is imported
Identification with a condensed phase spectrum	OPEN	Condensed spectrum remains a proposed construction
Connecting morphism in a pair	ESTABLISHED	Long exact sequence for generalized cohomology
Physical interpretation of every relative class	PROPOSED	Requires a microscopic invariant and natural comparison map

12 Limitations and open problems

12.1 Restricted model portfolio

All verified rows are one-dimensional. They do not test chiral phases in two spatial dimensions, intrinsic topological order, noninvertible phases, or higher-form symmetry. The AKLT row is a spin system, while the class-D and BDI rows are fermionic. Moving between these settings can involve a choice of graded tensor product or Jordan-Wigner transformation and is not treated as invisible.

12.2 No general field-theory emergence theorem

We do not prove that a gapped lattice model has a relativistic infrared limit, that its low-energy theory is fully extended, or that this theory is faithful to microscopic phase equivalence. These are among the assumptions needed to apply theorem 2.3 to lattice systems.

12.3 No surjectivity

Nothing here proves that every element of $[MTH, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tors}}$ is realized by a local Hamiltonian. Nothing proves that every homotopy class in a proposed condensed spectrum comes from Kubota's construction, or conversely.

12.4 Boundaries and degeneracy

Uniform bulk gaps and open-boundary spectral gaps are different statements. We explicitly retain the low-energy edge sector in equation (3). Boundary perturbations can split that sector while leaving the bulk gap open. A realizability row must state which notion it uses.

12.5 Relative charge needs orientation and comparison

The Thom identification in equation (23) can fail without an E-orientation of the normal bundle. Excision can fail if one works outside a category of good pairs without a supported-cohomology replacement. Most importantly, a relative generalized cohomology class is not automatically measurable until the microscopic invariant and response map are specified.

12.6 A useful next theorem

A major next step would be a natural transformation from a controlled microscopic invertible phase spectrum to an Anderson-dual bordism target, defined on smooth parameter families and compatible with stacking, suspension, spatial symmetry, and spectral flow. Its value on the three rows in this paper should recover the stated low-dimensional comparisons. Constructing that map is Open.

13 Conclusion

Physical realizability is a witness problem. A class earns microscopic status only after locality, symmetry, a thermodynamic gap, and an invariant have been checked on an explicit system. The Kitaev, BDI, and AKLT chains pass that test in one spatial dimension. Their agreement with spin, Pin^-, and $SO(3)$ field-theory data is informative and exact at the level stated, but it is not a universal realization theorem.

The broader condensed program therefore has a clear boundary. Condensed objects can organize parameterized witnesses and their descent. Kubota's Ω-spectrum supplies a genuine microscopic stable-homotopy object for quantum spin systems. Freed-Hopkins supplies a genuine bordism-based classification of reflection-positive invertible field theories. The maps between these constructions remain mathematical work to be done.

A Detailed witness sheets

A.1 Class D

Field	Witness
Spatial and spacetime dimensions	$d = 1$, $n = 2$
Local degrees of freedom	One complex fermion per site
Interaction	Nearest-neighbor hopping and pairing in equation (6)
Symmetry	Fermion parity; BdG particle-hole structure with square $+1$
Bulk boundary convention	Periodic chain
Edge convention	Open chain
Gap	Exact $2t$ at $\mu = 0$, $\Delta = t > 0$
Invariant	Pfaffian parity $\nu_D = -1$
Edge diagnostic	One unpaired Majorana at each end
Microscopic status	REALIZED
Abstract comparison	Nontrivial Arf/spin $\mathbb{Z}_2$ class under low-energy interpretation
Global comparison theorem	OPEN

A.2 BDI

Field	Witness
Spatial and spacetime dimensions	$d = 1$, $n = 2$
Local degrees of freedom	r real copies of the Kitaev chain
Interaction	Stacked quadratic chain plus allowed local quartic interactions
Symmetry	Antiunitary $T^2 = +1$ and fermion parity
Free bulk gap	Exact $2t$ at the stacked solvable point
Interaction witness	Fidkowski-Kitaev gapped trivialization of eight copies
Free invariant	$r \in \mathbb{Z}$
Interacting invariant	$r \bmod 8$

Microscopic status	REALIZED
Abstract comparison	Pin^- Arf-Brown-Kervaire $\mathbb{Z}_8$ agreement
Spectrum equivalence	OPEN

A.3 AKLT

Field	Witness
Spatial and spacetime dimensions	$d = 1$, $n = 2$
Local degrees of freedom	Spin one per site
Interaction	Nearest-neighbor total-spin-two projectors
Symmetry	Diagonal linear $SO(3)$ action
Bulk convention	Unique periodic ground state
Edge convention	Four-dimensional open-chain ground sector
Gap	Rigorous AKLT bulk-gap theorem
Invariant	Nontrivial class in $H^2(SO(3), U(1))$
Edge diagnostic	Projective spin-1/2 representation at each end
Microscopic status	REALIZED
Abstract comparison	Nontrivial bosonic $SO(3)$ field-theory sector
General realization theorem	OPEN

B A direct check of the AKLT projector

Let $x = \mathbf{S}_j \cdot \mathbf{S}_{j+1}$. For two spin-one variables,

$$x = \frac{1}{2}\left(J(J+1) - 2 \cdot 1(1+1)\right) \tag{26}$$

in the total-spin-J sector. Hence $x = -2, -1, 1$ for $J = 0, 1, 2$. The polynomial

$$p(x) = \frac{(x+2)(x+1)}{6} \tag{27}$$

has values $0, 0, 1$ on those sectors. Functional calculus therefore gives $p(x) = P^{(2)}$, proving equation (16). Positivity of every term and annihilation of the valence-bond state follow immediately. The spectral gap remains a separate theorem because a sum of positive local projectors need not be uniformly gapped.

C Relative-degree bookkeeping

For reference, the exact sequence and Thom shifts are

$$E^q(B) \xrightarrow{j^*} E^q(U) \xrightarrow{\partial} E^{q+1}(B,U) \xrightarrow{i^*} E^{q+1}(B) \tag{28}$$

and, for an E-oriented normal bundle of rank c,

$$E^{q+1}(B,U) \xrightarrow[\cong]{\text{excision}} E^{q+1}(D\mathcal{N}, S\mathcal{N}) \xrightarrow[\cong]{u_E^{-1}} E^{q+1-c}(\Sigma). \tag{29}$$

If $\nu = j^*\widetilde{\nu}$, then $\partial\nu = \partial j^*\widetilde{\nu} = 0$. If the normal bundle is not E-oriented, the final group must be replaced by the correctly twisted group. If Σ is singular, one needs a stratified normal datum or a supported-cohomology formulation rather than the smooth Thom statement used here.

D Machine-checkable registry contract

The finite registry validates the following implications:

$$\begin{aligned}
\texttt{REALIZED} &\Longrightarrow \texttt{interaction} \wedge \texttt{symmetry}, && (30)\\
&\qquad\qquad \texttt{gap witness} \wedge \texttt{microscopic invariant}, && (31)\\
\texttt{dimension valid} &\Longleftrightarrow n = d+1, && (32)\\
\texttt{extends globally} &\Longrightarrow Q_\Sigma(\nu) = 0, && (33)\\
\texttt{Thom degree} &= q+1-c \quad \text{when an orientation is present.} && (34)
\end{aligned}$$

It deliberately does not encode the false implication

$$\texttt{same finite group} \Longrightarrow \texttt{equivalent phase spectra}. \tag{35}$$

E Glossary of scope-sensitive terms

Term	Meaning in this paper
Microscopic phase	Stabilized uniformly gapped homotopy class of local interactions with fixed symmetry rules
Invertible	Has a stacking inverse up to stabilized gapped homotopy
Uniform gap	Positive lower bound independent of volume, with a declared low-energy sector
Abstract class	Element of a stated K-theory, bordism, homotopy, or generalized cohomology group

Realized	Complete microscopic witness for the specific claimed invariant
Candidate	Explicit model with at least one missing witness
Open	Neither realization nor obstruction is asserted
Obstructed	Nonrealizability follows from a theorem under the same rules
Relative class	Image under the connecting morphism in a long exact sequence
Physical charge	Relative class equipped with a microscopic comparison and response interpretation
Condensed family	Proposed family organized on condensed test objects with uniform analytic bounds

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